Stochastic Differential Equations, Part II

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Overview

- ▷ Stochastic differential equations
- ▷ Some methods
- ▷ Strong convergence
- ▷ Stability issues
- ▷ Remarks on weak methods

SODEs

Consider Itô stochastic ordinary differential equations (SODEs) on $[0,T]$

$$X(s)\Big|_0^t = \int_0^t f(s, X(s))\,\mathrm{d}s + \sum_{r=1}^{m} \int_0^t g_r(s, X(s))\,\mathrm{d}B_r(s),\;\; X(0) = X_0$$

with $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $G = (g_1, \ldots, g_m) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$. B is an m-dim. Brownian motion on a suitable probability space, X_0 is a given initial value, independent of the Brownian motion and with finite second moment, i.e. $\mathbb{E}|X_0|^2 < \infty$.

Numerics for Stochastic Ordinary Differential Equations

Integral equation on $t \in [0, T]$:

$$X(t) = x_0 + \int_0^t f(s, X(s))\mathrm{d}s + \int_0^t G(s, X(s))\mathrm{d}B(s)$$

Define:

Grid on $[0, T]$, step-size $h := T/N$ and $t_n = n \cdot h,\ n = 0, \ldots, N$,
$I^{t_n, t_{n+1}} = \int_{t_n}^{t_{n+1}} \mathrm{d}B(s) = B(t_{n+1}) - B(t_n) \sim \sqrt{h}\mathcal{N}(0, 1)$.

Simplest numerical method: Euler-Maruyama-method

$$X_{n+1} = X_n + h\, f(t_n, X_n) + G(t_n, X_n)\, I^{t_n, t_{n+1}}$$

Numerical methods

A basic numerical method: Θ-Maruyama scheme
grid on $[0, T]$ with step-size $h := T/N$ and $t_n = n \cdot h, \; n = 1, \ldots, N-1$

$$\begin{aligned} X_n &= X_{n-1} + h\big(\Theta\, f(t_n, X_n) + (1-\Theta)\, f(t_{n-1}, X_{n-1})\big) \\ &+ \sum_{r=1}^{m} g_r(t_{n-1}, X_{n-1})\, I_r^{t_{n-1}, t_n} \end{aligned}$$

where

$$I_r^{t_{n-1}, t_n} = B_r(t_n) - B_r(t_{n-1}) \sim \mathcal{N}(0, h)$$

$\Theta = 0$: Euler-Maruyama method

The Milstein method

$$
\begin{aligned}
X_n &= X_n + h\, f(t_{n-1}, X_{n-1}) + \sum_{r=1}^{m} g_r(t_{n-1}, X_{n-1})\, I_r^{t_{n-1},t_n}\,, \\
&+ \sum_{r_1,r_2=1}^{m} (g_{r_1})'_x\, g_{r_2}(t_{n-1}, X(t_{n-1}))\; I_{r_1,r_2}^{t_{n-1},t_n}
\end{aligned}
$$

where again

$I_r^{t_{n-1},t_n} = B_r(t_n) - B_r(t_{n-1}) \sim \mathcal{N}(0,h)$

and also

$I_{r_1,r_2}^{t_{n-1},t_n} = \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{s} \mathrm{d}B_{r_1}(u)\mathrm{d}B_{r_2}(s)$

Generation of paths of Brownian motion

Construction: $B(t_n) = B(0) + \sum_{i=0}^{n-1} (B(t_{i+1}) - B(t_i))$

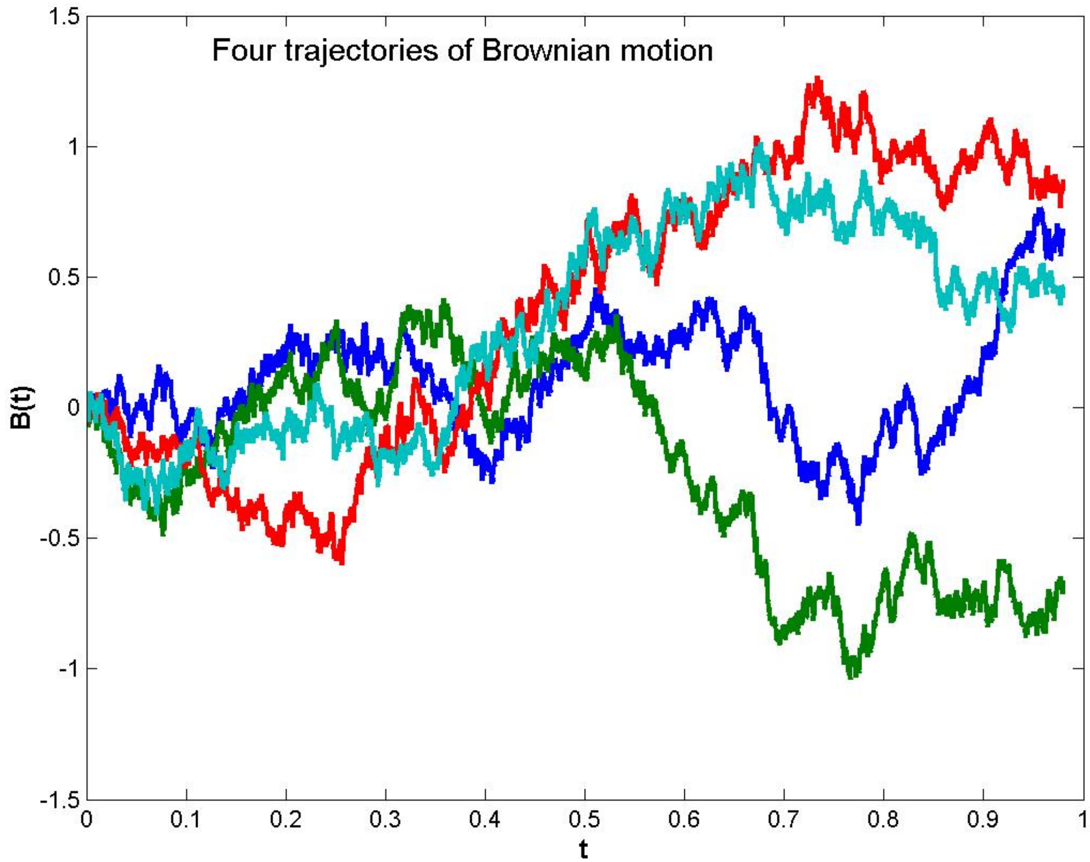
where $B(t) - B(s)$ is a $\sqrt{t-s}\mathcal{N}(0,1)$ distr. random variable, $0 \leq s < t \leq T$,

$\mathcal{N}(0,1)$ either comes from e.g., matlab, R, etc or can be calculated from uniform r.v. over $(0,1)$ (probably currently best: Mersenne twistor)

Polar-Marsaglia:

```
 generate(double *g1, double *g2) {
  double v1,v2,w,lw;

  do
     { v1 = 2.0 * frand()-1.0;
       v2 = 2.0 * frand()-1.0;
       w  = v1 * v1 + v2 * v2;
     }
  while (!((w<=1.0) && (w>0.0))); /* obviously ln(0) does not exist
  lw = log(w)/w;
  lw = sqrt(-lw-lw);
  *g1 = v1*lw;
  *g2 = v2*lw;
}
```



Four trajectories of Brownian motion
B(t)
t
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Questions:

How do we understand the quality of the approximations?

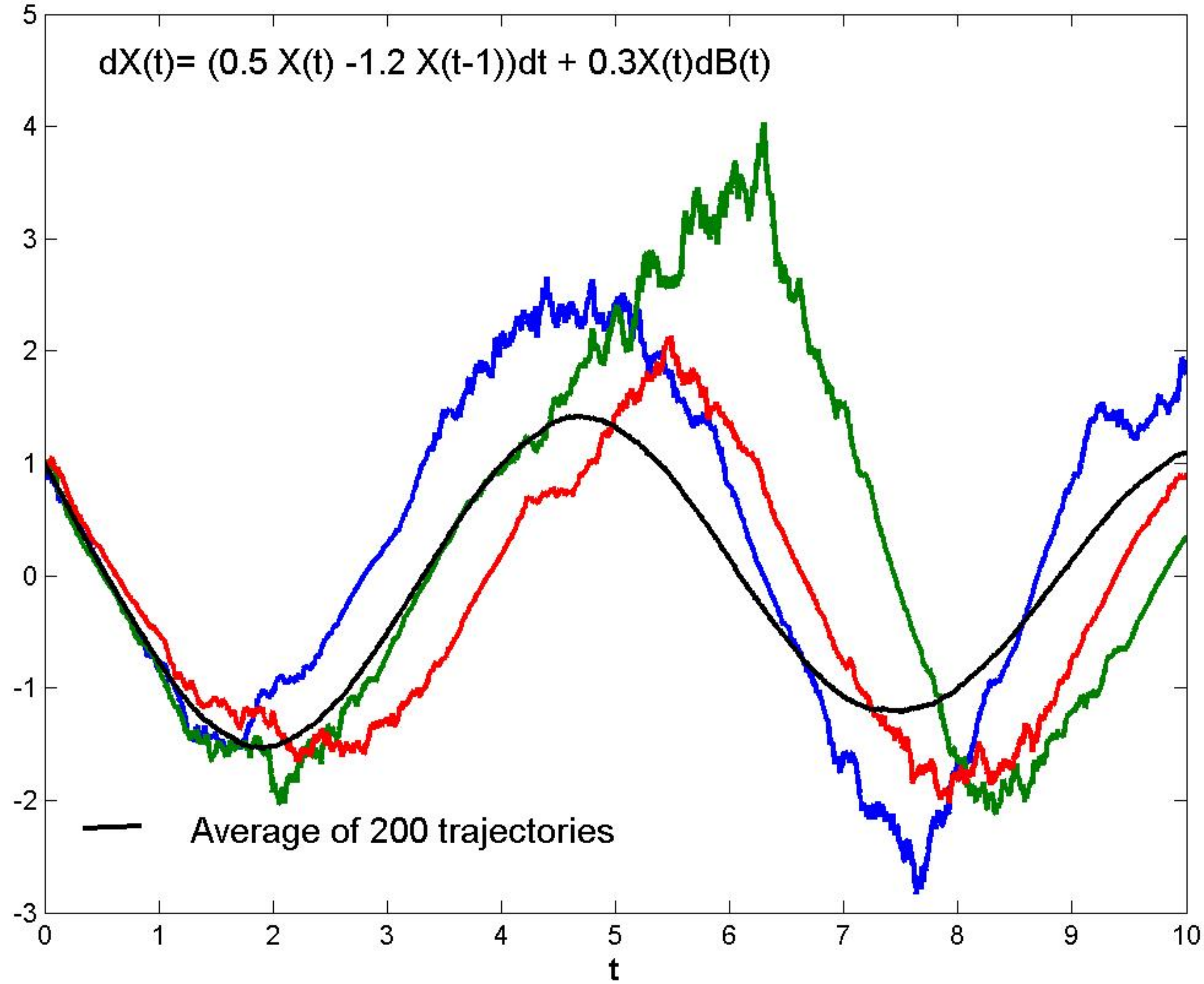
How do we analyse the quality?

How do we construct methods?

Two Modes of Convergence

- Strong convergence measures the (mean-square) difference of *paths* of the analytical and numerical solution; simulations used for visualisation of dynamics, filtering.

- Weak approximations are concerned with calculating the expectation of some functional of the solution $\mathbb{E}\,\Psi(X(t_n))$ where Ψ could be the moments of the process,
concerned with distributional properties of the processes, Monte-Carlo-methods,


dX(t)= (0.5 X(t) -1.2 X(t-1))dt + 0.3X(t)dB(t)
Average of 200 trajectories
t
5
4
3
2
1
0
-1
-2
-3
0
1
2
3
4
5
6
7
8
9
10

One-step methods

Let Y_n denote the numerical approximation of the exact solution $X(t_n)$ of an Itô SODE at time t_n on the given grid, where Y_n is given by a stochastic explicit one–step scheme of the form

$$Y_{n+1} = Y_n + \Phi(Y_n, h_n, t_n, I^{t_n, t_n+h_n}), \quad n = 0, \ldots, N-1$$

with initial value $Y_0 = X_0$. We call the function Φ the increment function of the method. The term $I^{t,t+h}$ denotes a collection of multiple stochastic integrals over the subinterval $[t, t+h]$

$$I^{t,t+h}_{r_1,r_2,\ldots,r_j} = \int_t^{t+h} \int_t^{s_1} \ldots \int_t^{s_{j-1}} \mathrm{d}B_{r_1}(s_j) \ldots \mathrm{d}B_{r_j}(s_1),$$

where $r_i \in \{0, 1, \ldots, m\}$ and $\mathrm{d}B_0(s) = \mathrm{d}s$.

Strong Convergence

We call a numerical method **mean-square convergent** if the global error $X(t_n) - X_n$ satisfies

$$\max_{n=1,\ldots,N} \|X(t_n) - Y_n\|_{L_2} \to 0 \text{ as } h \to 0,$$

we say the method is **mean-square convergent with order γ** ($\gamma > 0$) if the global error satisfies

$$\max_{n=1,\ldots,N} \|X(t_n) - Y_n\|_{L_2} \leq C \cdot h^{\gamma} \text{ as } h \to 0$$

($C > 0$ indep. of h).

Norm $\|Z\|_{L_2} := (\mathbb{E}|Z|^2)^{1/2}$.

Define mean-square consistency via the local error

$$L_n = X(t_{n+1}) - X(t_n) - \Phi(X(t_n), h_n, t_n, I^{t_n, t_n+h}), \quad n = 0, \ldots, N-1$$

of the method, measured in the mean ($\|\mathbb{E}(L_n|\mathcal{F}_{t_n})\|_{L_2}$) and in the mean-square ($\|L_n\|_{L_2}$).

Consistency measures the amount by which the numerical method fails to produce the true solution over one (small) interval $[t_n, t_n + h]$. The measurement in the mean ensures that the deterministic part of the SODE is approximated appropriately.

One can show that under Lipschitz conditions in the mean and the mean-square on the increment function Φ, that is

$$\textbf{(Lip1)} \quad \left|\mathbb{E}\big(\Phi(u_1,h,t,I^{t,t+h}) - \Phi(u_2,h,t,I^{t,t+h})\big|\mathcal{F}_t\big)\right| \leq Lh|u_1-u_2|,$$

$$\textbf{(Lip2)} \quad \mathbb{E}\Big(\big|\Phi(u_1,h,t,I^{t,t+h}) - \Phi(u_2,h,t,I^{t,t+h})\big|^2\Big|\mathcal{F}_t\Big) \leq L^2h|u_1-u_2|^2.$$

mean-square consistency of order γ implies mean-square convergence of the same order. Thus, we need to check two Lipschitz conditions and estimate the local error, using the Itô formula and a 'counting lemma'

Lemma (Milstein, Tretyakov): For any function ϕ sufficiently smooth, and any t on the grid, and step-size $h > 0$, such that $t+h$ is also on the grid, we have that

$$\begin{aligned}\mathbb{E}(I^{t,t+h}_{r_1 \ldots r_j}(\phi)|\mathcal{F}_t) &= 0 \quad \text{if } r_i \neq 0 \quad \text{for some } i \in \{1, \ldots, j\}, \\ \|\mathbb{E}(I^{t,t+h}_{r_1,\ldots,r_j}(\phi)|\mathcal{F}_t)\|_{L_2} &\leq \|I^{t,t+h}_{r_1,\ldots,r_j}(\phi)\|_{L_2} = \mathcal{O}(h^{l_1+l_2/2}),\end{aligned}$$

where l_1 is the number of zero indices r_i, and l_2 the number of nonzero indices r_i.

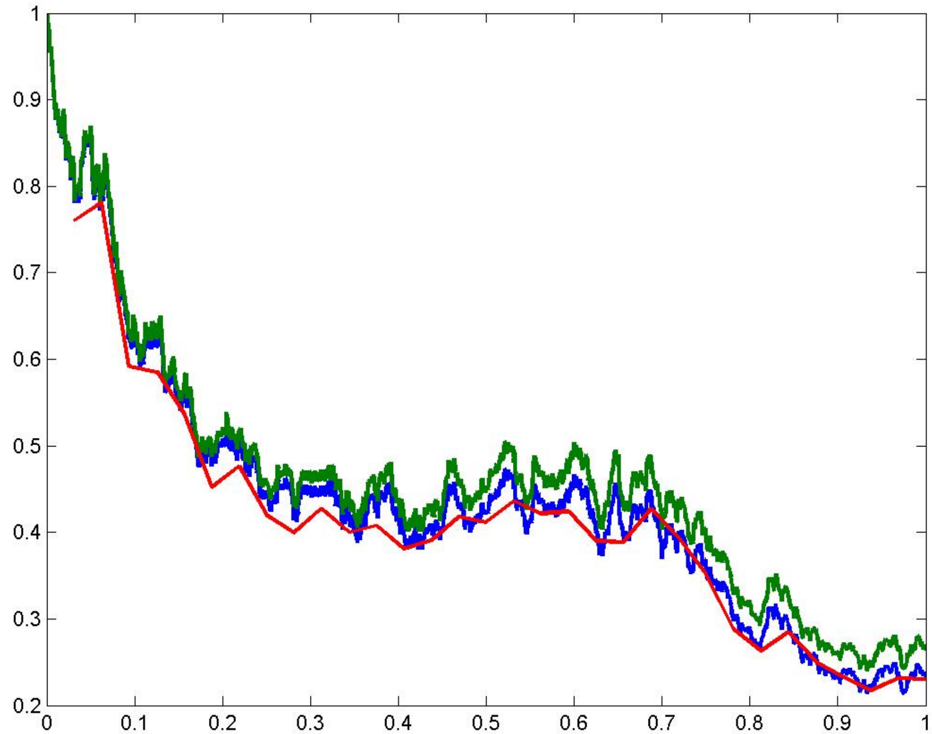
What happens for Stratonovich equations?

If we apply the Euler-Maruyama method

$$X_{n+1} \;=\; X_n \;+ h\, f(t_n, X_n) + G(t_n, X_n)\, I^{t_n, t_{n+1}}$$

to

$$\mathrm{d}X(t) = f(t, X(t))\ \mathrm{d}t + g(t, X(t)) \circ \mathrm{d}B(t)$$



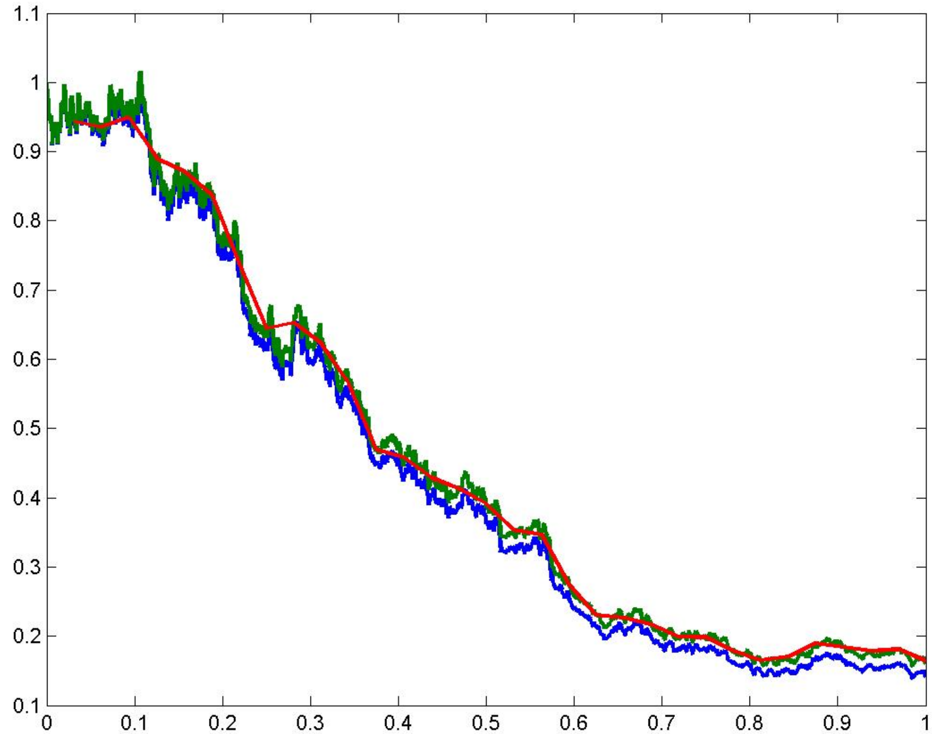
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Need to be consistent with Stratonovich calculus!

Try Heun-method

$$X_{n+1} \;=\; X_n \;+ h\, f(t_n, X_n) + \tfrac{1}{2}(G(t_n, X_n) + G(t_n, \tilde{X}_n))\, I^{t_n, t_{n+1}}$$

where $\tilde{X}_{n+1} \;=\; X_n \;+ h\, f(t_n, X_n) + G(t_n, X_n)\, I^{t_n, t_{n+1}}$


1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

are there other methods?

SODEs with small noise

Itô stochastic ordinary differential equations (SODEs) on $\mathcal{J} := [0, T]$

$$X(s)\Big|_0^t = \int_0^t f(s, X(s))\,\mathrm{d}s + \epsilon \int_0^t G(s, X(s))\,\mathrm{d}W(s), \quad X(0) = X_0$$

with $f : \mathcal{J} \times \mathbb{R}^n \to \mathbb{R}^n$ and $G = (g_1, \ldots, g_m) : \mathcal{J} \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$. W is an m-dim. Wiener process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathcal{J}}, \mathbb{P})$, X_0 is a given $\mathcal{F}_0$-measurable initial value, independent of the Wiener process and with finite second moment. $\epsilon \ll 1$!

We assume that there exists a path-wise unique strong solution $X(\cdot)$ of the above equation.

Stochastic linear two-step Maruyama method (SL2MM)

for $n = 2, 3, \ldots$ with given i.v. $X_0,\ X_1 \in L_2(\Omega, \mathbb{R}^n)$

$$\begin{aligned} \sum_{j=0}^{2} \alpha_{2-j}\, X_{n-j} &= h \sum_{j=0}^{2} \beta_{2-j}\, f(t_{n-j}, X_{n-j}) \\ &+ \sum_{j=1}^{2} \gamma_j \sum_{r=1}^{m} g(t_{n-j}, X_{n-j})\, I_r^{t_{n-j}, t_{n-j+1}} \end{aligned}$$

mean-square convergence theory in B., Winkler, SINUM, 2006

$\alpha_2 = 1$, $\gamma_1 = \alpha_2 = 1$ and $\gamma_2 = \alpha_2 + \alpha_1$

E.g.: BDF2-Maruyama method: $\alpha_2 = 1$, $\alpha_1 = -\frac{4}{3}$, $\alpha_0 = \frac{1}{3}$, $\beta_2 = \frac{2}{3}$, $\beta_1 = \beta_0 = 0$, $\gamma_1 = 1$ and $\gamma_2 = -\frac{1}{3}$

Numerical results

$X(t) = 1 + \int_0^t \alpha X(s)ds + \int_0^t \beta X(s)dW(s),\ t \in [0,1],$ (scalar)

with solution (gBm) $X(t) = \exp\big((\alpha - \frac{1}{2}\beta^2)t + \beta W(t)\big)$

maximum approximate L_2-norm of the global errors on $[0,1]$:

$$\max_{\ell=1,\ldots,N} \Big(\frac{1}{M}\sum_{j=1}^{M} |X(t_\ell,\omega_j) - X_\ell(\omega_j)|^2\Big)^{1/2} \approx \max_{\ell=1,\ldots,N} \|X(t_\ell) - X_\ell\|_{L_2}$$

N $\equiv$ number of steps, $M \equiv$ number of computed paths ($M = 100$)

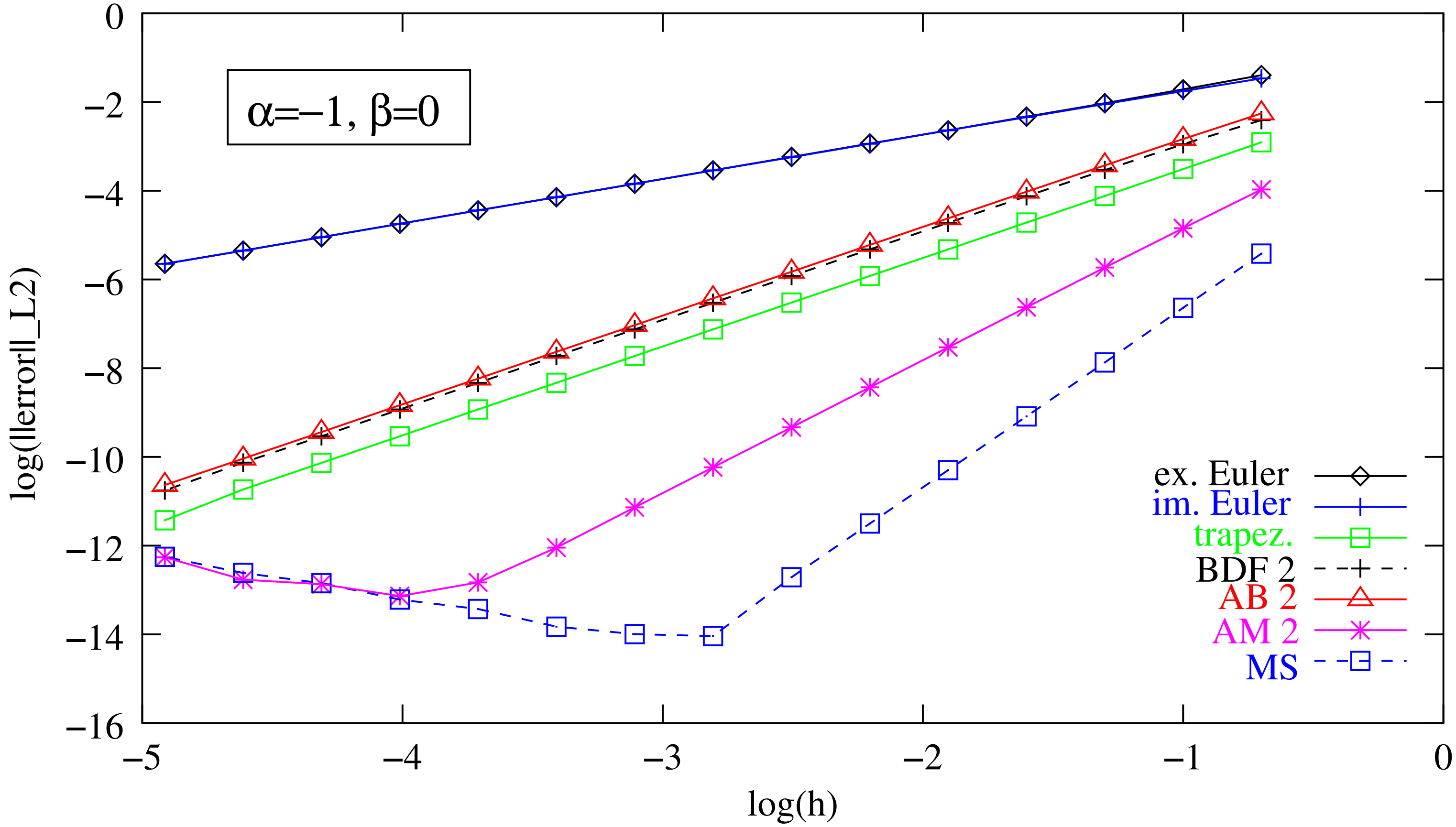

α=−1, β=0
0
−2
−4
−6
−8
−10
−12
−14
−16
−5
−4
−3
−2
−1
0
log(||error||_L2)
log(h)
ex. Euler
im. Euler
trapez.
BDF 2
AB 2
AM 2
MS

Figure 2

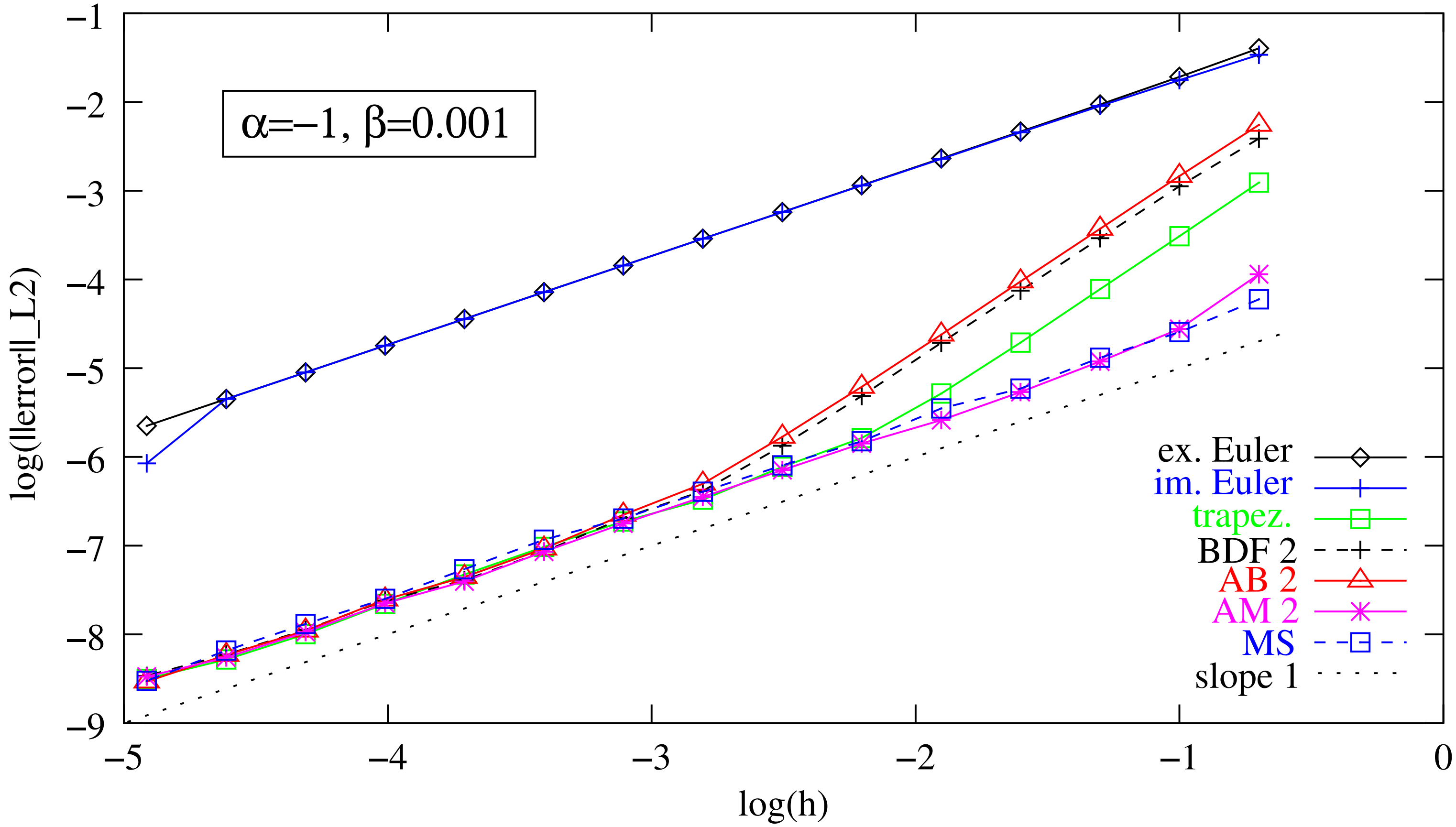
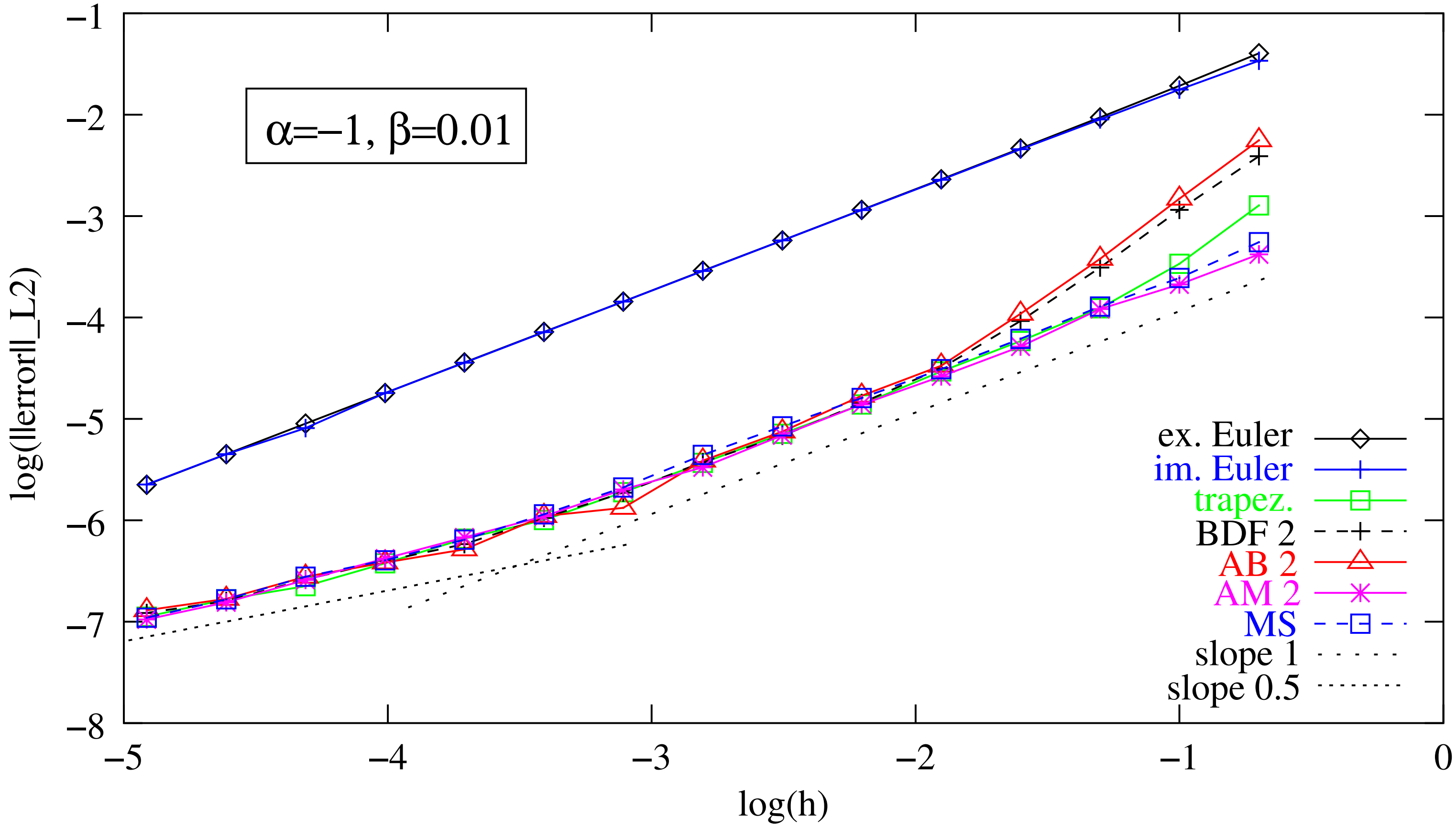
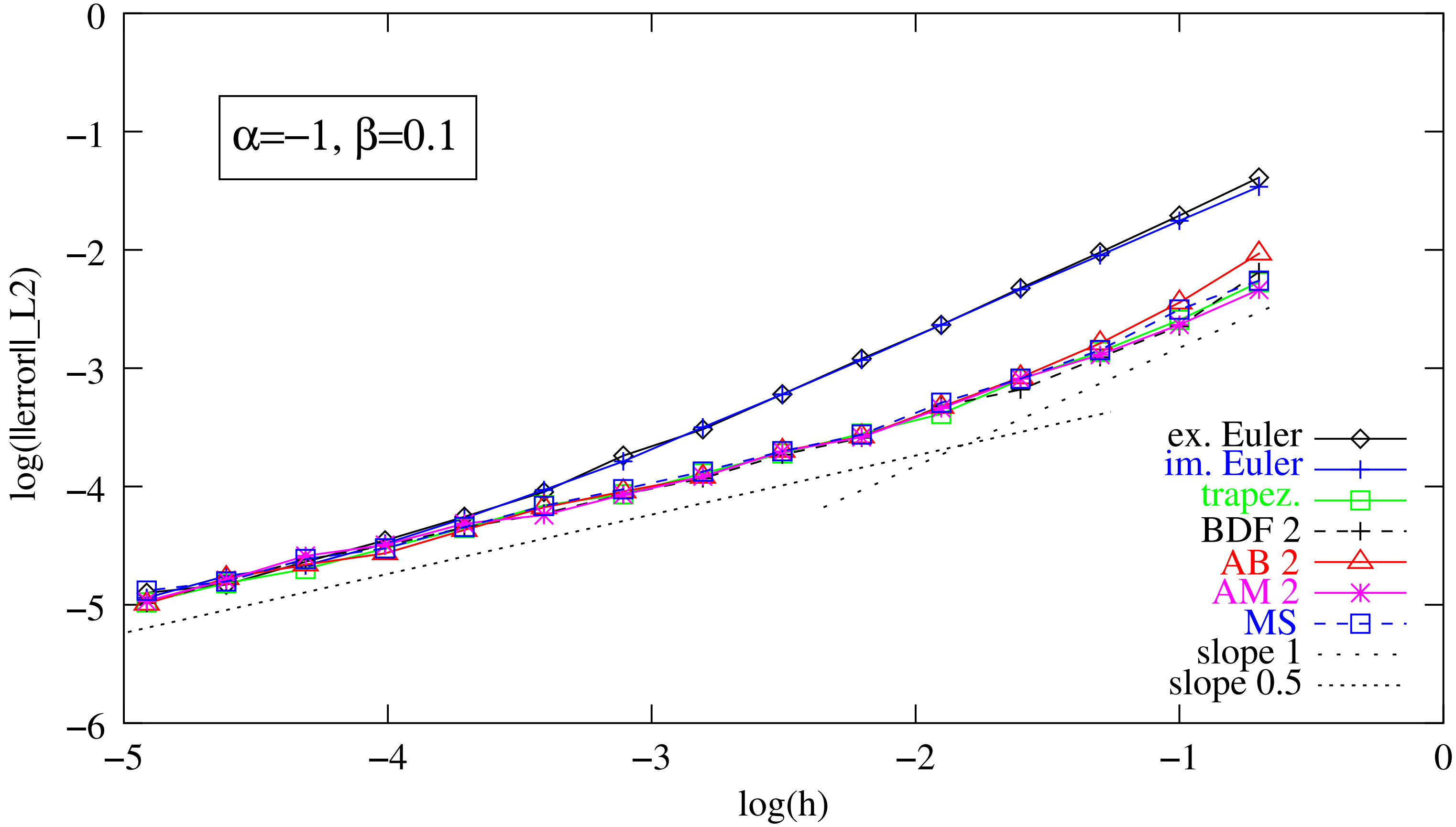

α=−1, β=0.001
log(||error||_L2)
log(h)
−1
−2
−3
−4
−5
−6
−7
−8
−9
−5
−4
−3
−2
−1
0
ex. Euler
im. Euler
trapez.
BDF 2
AB 2
AM 2
MS
slope 1

Figure 3



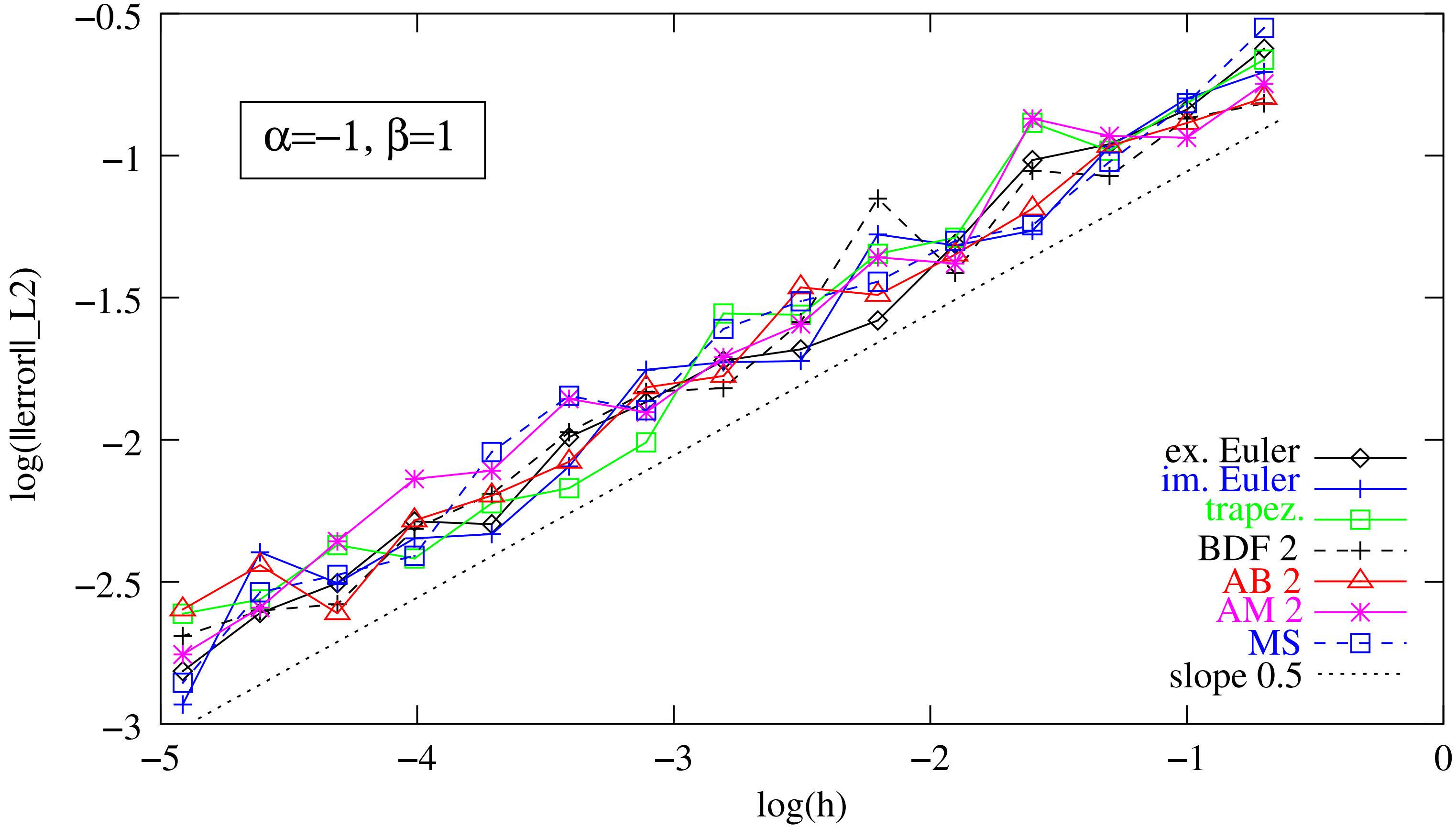
α=−1, β=0.01
log(||error||_L2)
log(h)
−1
−2
−3
−4
−5
−6
−7
−8
−5
−4
−3
−2
−1
0
ex. Euler
im. Euler
trapez.
BDF 2
AB 2
AM 2
MS
slope 1
slope 0.5

Figure 4



α=−1, β=0.1
0
−1
−2
−3
−4
−5
−6
−5
−4
−3
−2
−1
0
log(||error||_L2)
log(h)
ex. Euler
im. Euler
trapez.
BDF 2
AB 2
AM 2
MS
slope 1
slope 0.5

Figure 5



−0.5
−1
−1.5
−2
−2.5
−3
−5
−4
−3
−2
−1
0
log(||error||_L2)
log(h)
α=−1, β=1
ex. Euler
im. Euler
trapez.
BDF 2
AB 2
AM 2
MS
slope 0.5

Figure 6

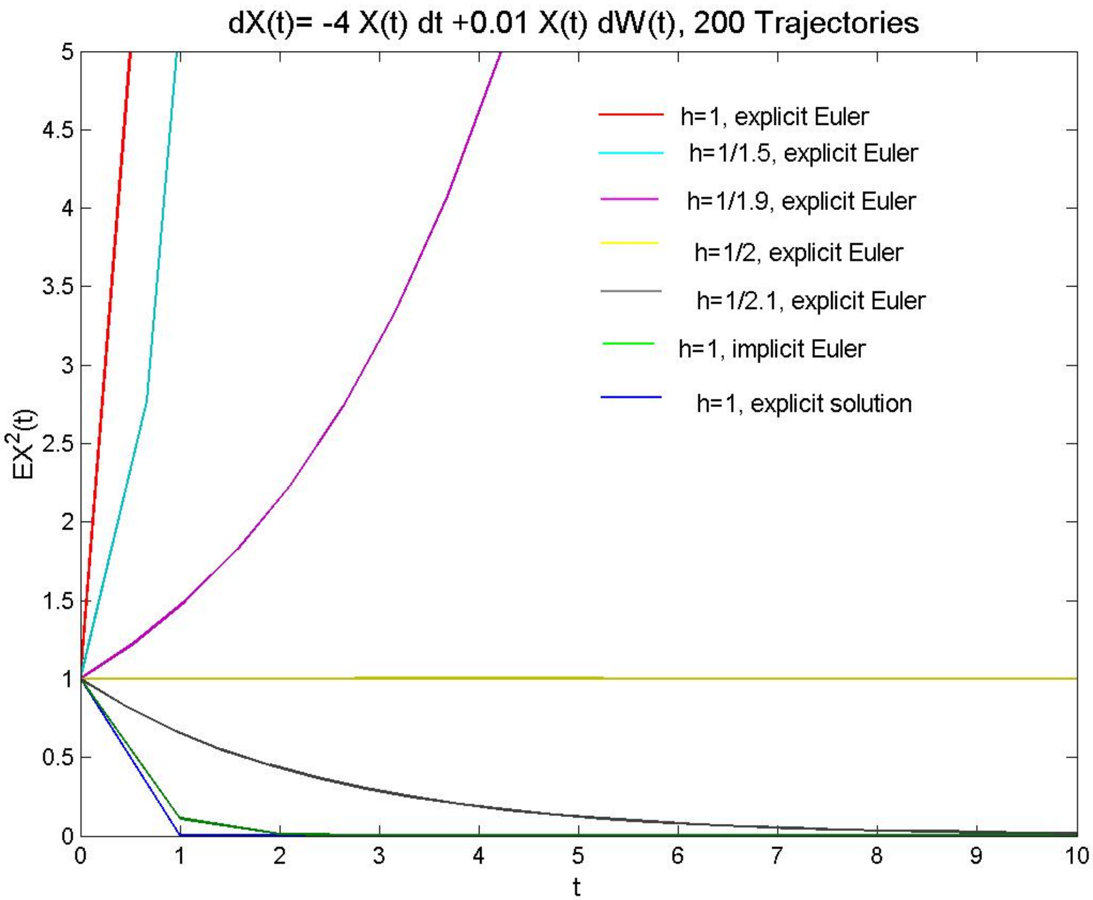
SRK-Maruyama

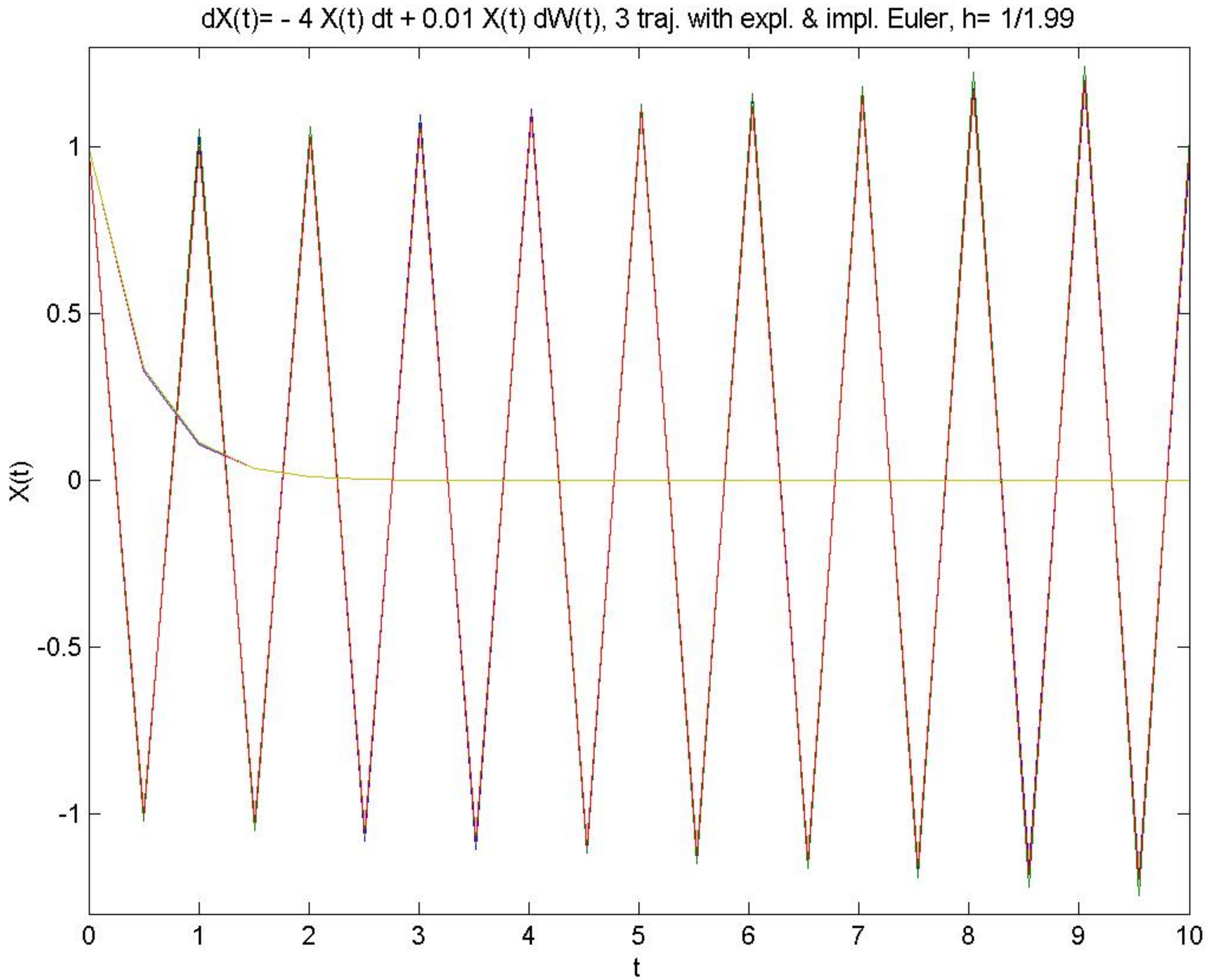
Grid on $\mathcal{J}$ with constant step-size h and $t_n = n \cdot h,\ n = 0,\ 1,\ 2, \ldots, N$.

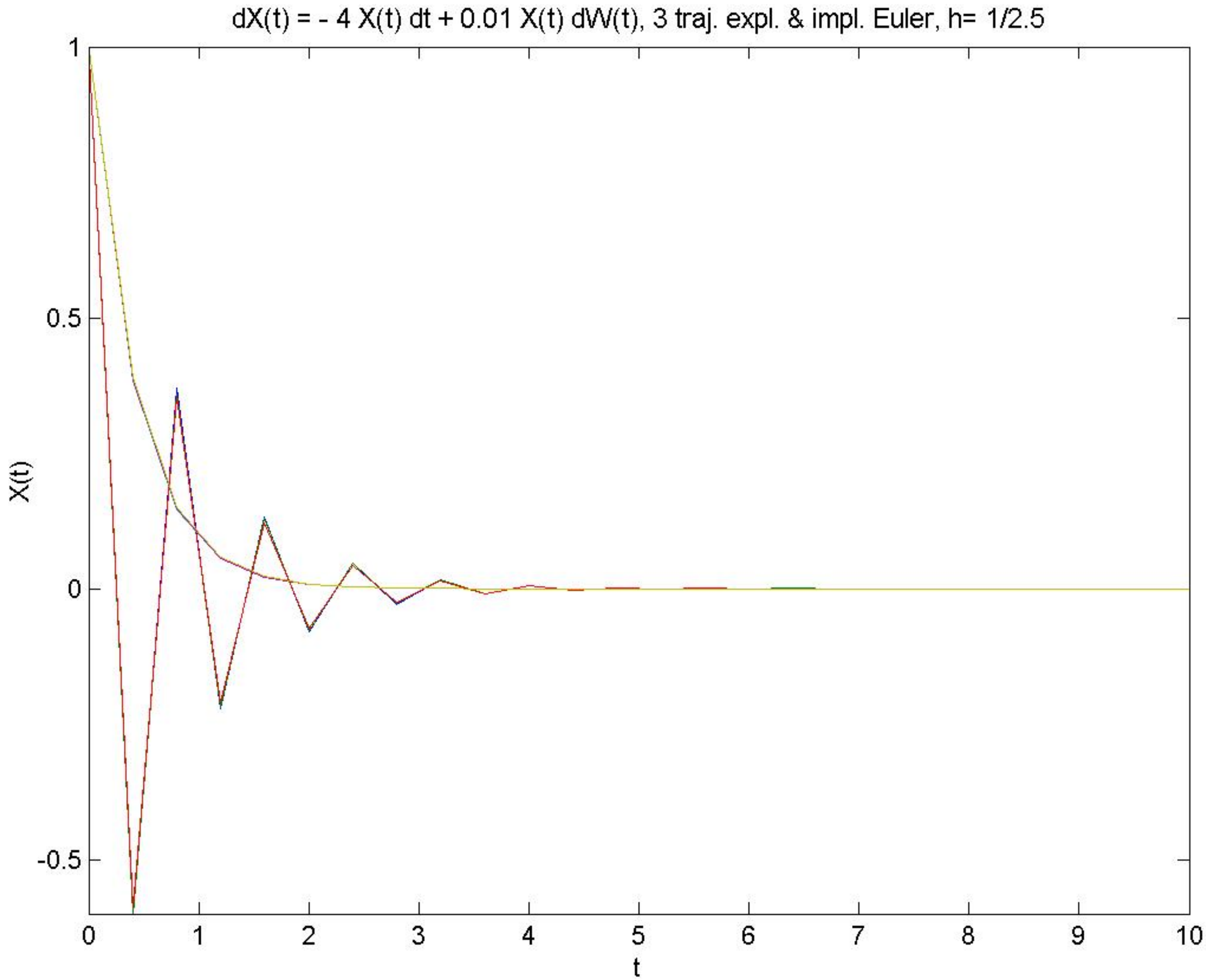
$$
\begin{aligned}
Y_0 &= X_0, \\
Y_{n+1} &= Y_n + \sum_{i=1}^{s} \beta_i\, f(t_n + c_i h, H_i)\, h + \sum_{r=1}^{m} g_r(t_n, Y_n)\, I_r^{t_n, t_n+h}, \\
H_i &= Y_n + \sum_{j=1}^{s} a_{ij} f(t_n + c_j h, H_j)\, h, \quad i = 1, \ldots, s
\end{aligned}
$$

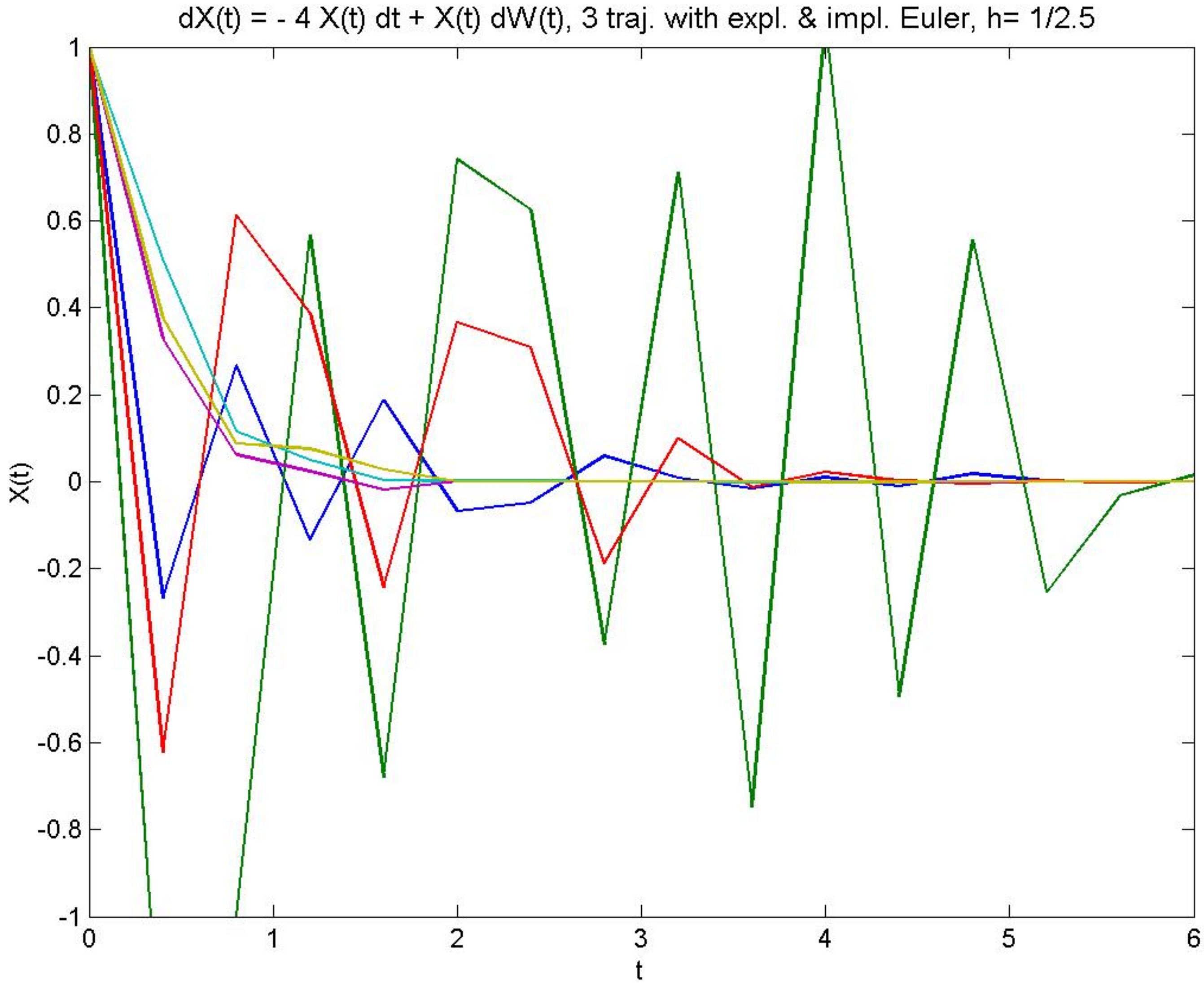
$$I_r^{t_n, t_{n+1}} = B_r(t_{n+1}) - B_r(t_n) \sim \mathcal{N}(0, h)$$

Stability behaviour




dX(t)= - 4 X(t) dt + 0.01 X(t) dW(t), 3 traj. with expl. & impl. Euler, h= 1/1.99
X(t)
t
1
0.5
0
-0.5
-1
0
1
2
3
4
5
6
7
8
9
10


dX(t) = - 4 X(t) dt + 0.01 X(t) dW(t), 3 traj. expl. & impl. Euler, h= 1/2.5
X(t)
t
1
0.5
0
-0.5
0
1
2
3
4
5
6
7
8
9
10


dX(t) = - 4 X(t) dt + X(t) dW(t), 3 traj. with expl. & impl. Euler, h= 1/2.5
X(t)
t
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
1
2
3
4
5
6

Asymptotic MS-stability of zero solutions

Def.: The zero solution of the SODE is termed

- *mean-square stable*, if for each $\epsilon > 0$, there exists a $\delta \geq 0$ such that the solution $X(t) = X(t; 0, X_0)$ exists for all $t \geq 0$ and $\mathbb{E}|X(t)|^2 < \epsilon$ whenever $t \geq 0$ and $\mathbb{E}|X_0|^2 < \delta$;

- *asymptotically mean-square stable*, if it is mean-square stable and if there exists a $\delta \geq 0$ such that whenever $\mathbb{E}|X_0|^2 < \delta$ we have $\mathbb{E}|X(t)|^2 \to 0 \quad \text{for} \quad t \to \infty$.

The definition for the zero solution of the numerical method is analogous.

Linear test equation, for $t \geq 0$, with $X(0) = X_0, \quad \lambda, \mu, X_0 \in \mathbb{C}$,

$$\mathrm{d}X(t) = \lambda\, X(t)\mathrm{d}t \;+\; \mu\, X(t)\mathrm{d}B(t), \qquad (1)$$

with the complex geometric Brownian motion as exact solution.

Thm.: (e.g. in Arnold '74, Hasminskii '80)
The zero solution of (1) is asymptotically mean-square stable if

$$Re(\lambda) < -\tfrac{1}{2}\, |\mu|^2$$

Linear stability analysis of Θ-method (D. Higham)

$$\begin{aligned} X_n &= X_{n-1} + h\big(\Theta\, f(t_n, X_n) + (1-\Theta)\, f(t_{n-1}, X_{n-1})\big) \\ &+ \sum_{r=1}^{m} g_r(t_{n-1}, X_{n-1})\, I_r^{t_{n-1},t_n} \end{aligned}$$

applied to $\mathrm{d}X(t) = \lambda\, X(t)\mathrm{d}t \;+\; \mu\, X(t)\mathrm{d}B(t)$ gives recurrence

$$X_{i+1} = (\widetilde{a} + \widetilde{b}\,\xi_i)\;X_i, \quad \text{where} \quad \widetilde{a} = \frac{1 + (1-\theta)\lambda h}{1-\theta\lambda h}, \quad \widetilde{b} = \frac{\mu h^{\frac{1}{2}}}{1-\theta\lambda h}.$$

Squaring and taking the expectation $\Rightarrow$ *exact* one-step rec. for $\mathbb{E}|X_i|^2$

$$\mathbb{E}|X_{i+1}|^2 = (|\widetilde{a}|^2 + 2|\widetilde{a}|\,|\widetilde{b}|\,|\mathbb{E}\,\xi_i| + |\widetilde{b}|^2\,|\mathbb{E}\,\xi_i^2|)\;\mathbb{E}|X_i|^2 = (|\widetilde{a}|^2 + |\widetilde{b}|^2)\;\mathbb{E}|X_i|^2$$

This recurrence will converge to zero, that is its zero solution will be asymptotically mean-square stable, iff $(|\widetilde{a}|^2 + |\widetilde{b}|^2) < 1$.

Weak Convergence

Weak approximations are concerned with calculating the expectation of some functional of the solution $\mathbb{E}\,\Psi(X(t_n))$ where Ψ could be the moments of the process or, of course the price function of an option! The measure is concerned with distributional properties of the processes, thus convergence measures the difference of $|\mathbb{E}\,\Psi(X(t_n)) - \mathbb{E}\,\Psi(X_n)\,|$,
key-word: Monte-Carlo-methods.

(see Kloeden & Platen, Milstein & Tretyakov, Glasserman)

Euler and Milstein methods are the same methods!

Theorem(Milstein): if Ψ is suffiently smooth and polynmially bounded and for sufficiently large r the $2r$-th moments of the numerical approximation exist and are uniformly bounded wrt the number of time-steps, and, further, the one-step method under investigation satisfies

$$|\mathbb{E}(\Psi(X(t_n+h;t_n,x)) - \mathbb{E}\Psi(X_{n+1}(t_n,x)))| \leq K(x)h^{p+1}$$

where $X(t_n+h;t_n,x)$ denotes the solution of the SDE starting at time t with i.v. x, similarly for $X_{n+1}(t_n,x))$. Then

$$\max_{0\text{''}oeqn\leq} |\mathbb{E}\,\Psi(X(t_n)) - \mathbb{E}\,\Psi(X_n)\,| \leq ch^p$$

where ψ needs to be $2(p+1)$ times differentiable.

References

[1] L. Arnold. *Stochastic Differential Equations: Theory and Applications*. Wiley-Interscience, New York, 1974.

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