Fishery Management: the standard case

Saint Andrews February 26 2009

Introduction

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3. A singular calculus of variations problem.

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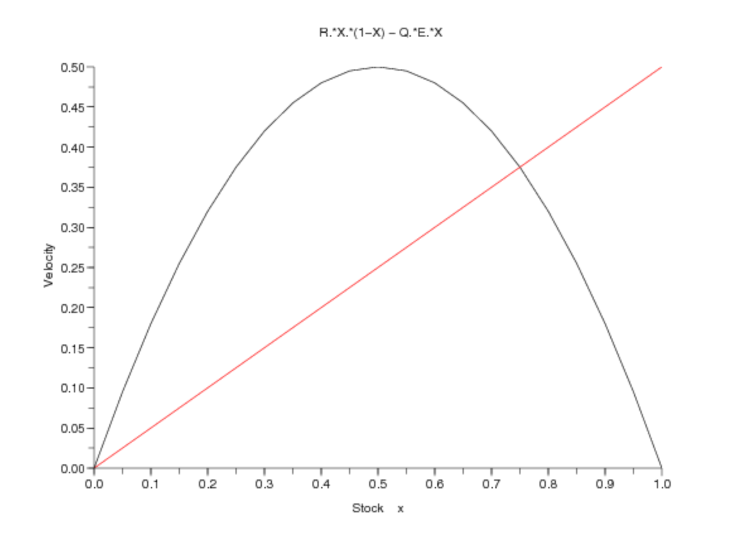
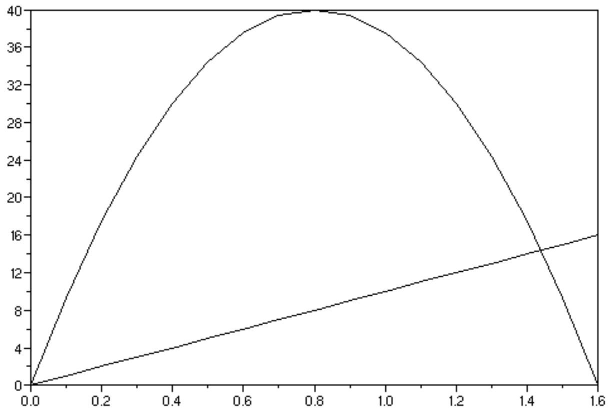
For stationary effort, $E(t) = E$, two equilibrium :

- $\bar{x} = 0$ unstable
- $\bar{x}$ stable

Graphic


R.*X.*(1−X) − Q.*E.*X
Velocity
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Stock x

Sustainability : $\bar{x}, \bar{E}$



$$\dot{x}(t) = rx(t)(1 - \frac{X(t)}{K}) - qEx(t)$$

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$$\max_{E(.)} \quad J(E(.), x_0)$$
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- $$Adm(x_0) := \{x(.) : [0, \infty] \to [0, K], x(0) = x_0,$$

$$\dot{x}(t) \in [f^-(x(t)) := f(x(t)) - qE_M x(t),\ f^+(x(t)) := f(x(t))\ ]\}$$

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$$\begin{array}{ll} l_x - \frac{d}{dt} l_{\dot{x}} + \delta l_{\dot{x}} & = 0 \\ A'(x) + B'(x)\dot{x} - \frac{d}{dt}[B(x(t))\dot{x}(t)] \quad +\delta B(x) & = 0 \end{array}$$

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Unique solution $\bar{x} \in (0, K)$

MRAP

Most Rapid Approach Path : MRAP$(x_0, \bar{x})$

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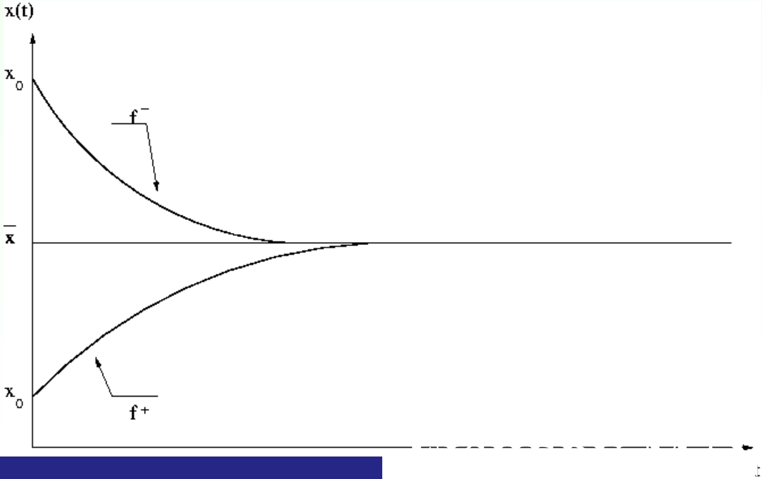

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- Hartl Feichtinger transversality condition :

$$\limsup_{t \to +\infty} [e^{-\delta t} \int_{x(t)}^{\bar{x}} B(\xi) d\xi] \geq 0$$

On the Green's theorem

$$\oint_{PQ} e^{-\delta t} A(x) dt + e^{-\delta t} B(x) dx = \int\int_{\mathcal{D}_i} e^{-\delta t} C(x) dt dx.$$

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If

$$\forall x, C(x)(\bar{x} - x) \geq 0$$

then

$$\text{MRAP}(x_0, \bar{x}) \text{ are optimal}$$

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Other Generalisations with the help of Value function approach and the Hamilton-Jacobi equation.