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Optimal Portfolios, Part I: Basic Methods in the Continuous-Time Setting

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1. Optimal Investment: A classical problem

In ancient times (3000 years ago):
First suggestions by the Babylonians (***Diversify into house, cash and production tools***)

In the literature:

Shakespeare ***The Merchant of Venice***
Cervantes ***Don Quijote*** (***Do not put all your eggs in one basket***)

In the fifties:

H. Markowitz ***Mean-Variance Approach***

Scientific state of the art:

Dynamic multi period models, martingale method, HJB equation, duality approaches, (quasi) variational inequalities, …

Practitioner's state of the art:

One-period models, variants of Markowitz ….

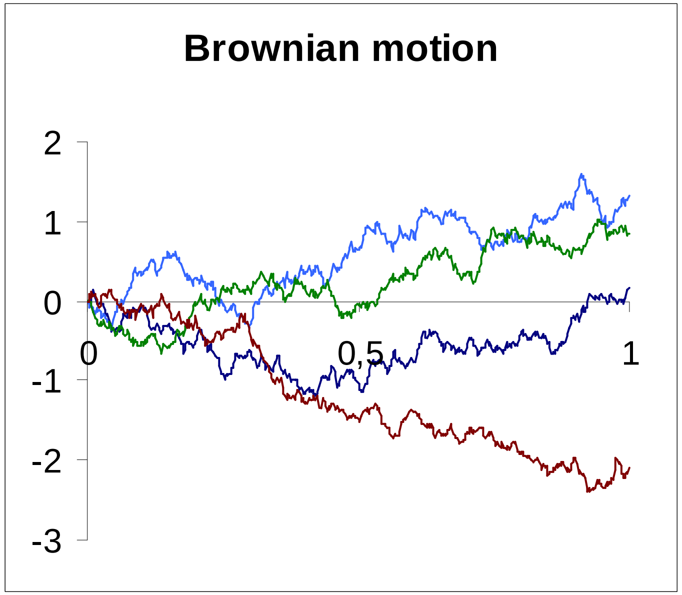
Aim of this mini-workshop: Present recent and applicable results and methods

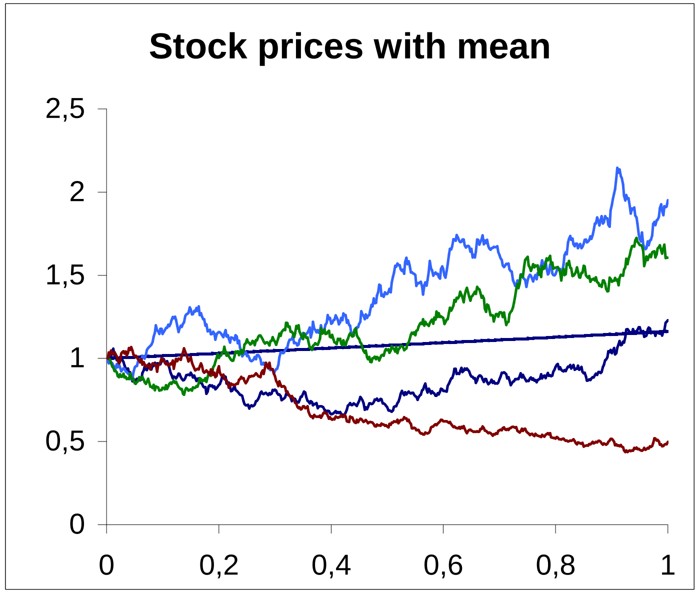
2. Market setting and the portfolio problem

The security prices (diffusion setting):

$$dP_0(t) = P_0(t)\, r(t)\, dt, \qquad P_0(0) = 1, \qquad \text{“}\textbf{\textit{Bond}}\text{”}$$

$$\mathrm{d}P_i(t) = P_i(t)\left(b_i(t)\,dt + \sum_{j=1}^{m} \sigma_{ij}(t)\, dW_j(t)\right), \quad i = 1,.., n, \qquad P_1(0) = p, \qquad \text{“}\textbf{\textit{Stocks}}\text{”}$$





The trading activities:

$\varphi_i(t)$: ***trading strategy*** (= no. of shares of security *i* that the investor holds at time *t*)

$c(t) \geq 0$: ***consumption rate process*** (= (velocity of) consumption at time *t*)

$X(t) := \sum_{i=0}^{n} \varphi_i(t) P_i(t)$: ***wealth process*** (= value of all holdings at time *t*)

Definition 1:
A pair (φ, c) of a trading strategy and a consumption rate process is a called ***self-financing strategy*** if we have

$$(1) \qquad X(t) = X(0) + \sum_{i=1}^{n} \int_0^t \varphi_i(s)\, dP_i(s) - \int_0^t c(s)\, ds$$

i.e. **wealth equals initial wealth plus gains / losses from investment minus consumption** .

Remark:
We assume that both processes (trading and consumption) are only based on past price observations (are “progressively measurable”) and satisfy suitable integrability conditions.

The wealth equation:

Introduce a ***portfolio process*** $\pi(t)$ (corresponding to a self-financing pair (φ, c)) as an n-dimensional stochastic process with components given by

(2) $$\pi_i(t) := \frac{\varphi_i(t) P_i(t)}{X(t)}, \quad i = 1, \ldots, n$$ "**fraction of wealth in stock *i***".

$\Rightarrow$ we obtain the following SDE ("***the wealth equation***") for the wealth process:

(3) $$dX(t) = \Big(X(t)\big(r(t) + \pi(t)'\big(b(t) - r(t)\underline{1}\big)\big) - c(t) \Big) dt + X(t)\pi(t)'\sigma(t) dW(t), \quad X(0) = x$$

Example: "*Linear strategies*"

$n = m = 1$, b, r, σ constant market coefficients and $\pi(t) = \pi, \quad c(t) = \gamma X(t)$:

(4) $$X(t) = x \cdot \exp\Big(\big(r + \pi(b - r) - \gamma - \tfrac{1}{2}\pi^2\sigma^2 \big) t + \pi\sigma W(t) \Big) > 0$$

Definition 2:

We will also call the pair (π, c) an ***admissible***, self-financing pair (and write $(\pi, c) \in A(x)$) if the corresponding wealth process stays non-negative after starting with an initial wealth of x.

Formulation of the portfolio problem:

Definition 3

i) A strictly concave C^1-function U: $(0, \infty) \to \mathbf{R}$ is called a ***utility function*** if it satisfies

(5) $$U'(0) := \lim_{x \downarrow 0} U'(x) = +\infty, \quad U'(\infty) := \lim_{x \to \infty} U'(x) = 0 \; .$$

ii) The (unconstrained) ***portfolio problem*** with initial wealth of x consists of solving

$$\boxed{\text{(P)} \qquad \max_{(\pi,c) \in A'(x)} \; E\left(\int_0^T U_1\big(t, c(t)\big) dt + U_2\big(X(T)\big) \right)}$$

with $A'(x) := \{(\pi,c) \in A(x) \mid E\left(\int_0^T U_1^-\big(t, c(t)\big) dt + U_2^-\big(X(T)\big) \right) < \infty\}$.

Examples of utility functions:

$$U(x) = ln(x), \qquad U(x) = \frac{1}{\gamma} x^\gamma, \; \gamma < 1, \qquad U(t,x) = e^{-\beta t} \frac{1}{\gamma} x^\gamma, \; \gamma < 1, \; \beta \geq 0$$

Properties of utility functions:

strictly increasing $\cong$ **more is always better than less**
concavity $\cong$ **decreasing marginal utility**, $\quad E\big(U(X)\big) \leq U\big(E(X)\big) \quad \cong$ **risk averse investor**

3. Optimal investment in complete markets: The martingale approach

The Fundamental Result:

Theorem "*Completeness of the market*"

Let $n = m$ and

(6) $$\theta(t) := \sigma(t)^{-1}(b(t) - r(t)\underline{1}), \quad H(t) := \exp\left(-\int_0^t \left(r(s) - \tfrac{1}{2}\|\theta(s)\|^2 \right) ds - \int_0^t \theta(s)' dW(s) \right)$$

a) For every $(\pi, c) \in A(x)$ we have

(7) $$E\left(H(t)X(t) + \int_0^t H(s)c(s)ds \right) \le x \, .$$

b) Let B be a contingent claim and $c(.)$ a consumption process with

(8) $$x := E\left(H(T)B + \int_0^T H(s)c(s)ds \right) < \infty \, .$$

Then there exists a portfolio process $\pi(.)$ such that we have $(\pi, c) \in A(x)$ and the corresponding wealth process $X(t)$ replicates the claim B, i.e. we obtain:

(9) $$X(T) = B \;\; a.s. \;\; .$$

Interpretation of the complete markets theorem:

Part a) yields:

Given a desired consumption process c(.) and a desired final wealth B then they are never realizable if we have

$$E\left(H(T)B + \int_0^T H(s)c(s)\,ds \right) > x$$

where x is the initial wealth of the investor.

Part b) yields:

Each desired consumption process c(.) and desired final wealth B can exactly be realized via following a suitable portfolio process π if we have an initial capital of

$$x := E\left(H(T)B + \int_0^T H(s)c(s)\,ds \right).$$

First consequence:

The unique fair price of a contingent claim with final payoff B is given by

$$E\left(H(T)B\right) \quad \left(= E_Q\left(e^{-rT}B\right)\right)$$

Main idea of the martingale approach (without consumption)**:**

Decompose the **dynamic portfolio problem**

(P) $$\max_{\pi \in A'(x)} E\left(U\left(X^{x,\pi}(T)\right)\right)$$

into a **static optimisation problem**

(O) $$\max_{B \in B(x)} E(U(B))$$

with $B(x) := \{B \mid B \geq 0,\ F_T\text{-meas.},\ E(H(T)B) \leq x,\ E\left(U(B)^{-}\right) < \infty\}$,

and a **representation problem**

“Find a portfolio process $\pi^* \in A'(x)$ with

(R) $$X^{x,\pi^*}(T) = B^* \quad \text{a.s. ”,}$$

where B* solves problem (O).

Step 1: ***Solution of the Optimisation Problem* (O)**

Proposition

Let $X(\mathrm{y}) := E\left(H(T) I_2\left(yH(T)\right) + \int_0^T H(t) I_1\left(t, H(t)\right) dt \right) < \infty \quad \forall\ \mathrm{y} > 0 \qquad (*)$

Then, X is continuous on $(0,\infty)$ and strictly decreasing with $X(\infty) = 0$, $X(0) = \infty$

Theorem 1

Let $x > 0$. Under assumption (*) the optimal terminal wealth B^* and the optimal consumption process $c^*(t)$, $t \in [0, T]$, are given by

(10) $\qquad B^* := I_2(Y(x)H(T)), \qquad c^*(\mathrm{t}) := I_1(t, Y(x)H(t))$,

and there exists a portfolio process $\pi^*(t)$, $t \in [0, T]$, such that we have

(11) $\qquad (\pi^*, c^*) \in A'(x)$, $\quad X^{x,\pi^*,c^*}(T) = \mathrm{B}^*$ a.s., $\quad J(x^*, \pi^*, c^*) = \max\limits_{(\pi,c)\in A'(x)} J(y\,;\pi, c)$,

i.e. (π*, c*) solves the unconstrained portfolio problem.

Corollary

Assume that the conditions of Theorem 1 are satisfied.

a) The optimal consumption process $c^*(t)$, $t \in [0, T]$, for the ***consumption problem***

$$\max_{(\pi, c) \in A'(x)} E\left(\int_0^T U_1(t, c(t) dt) \right)$$

is given by

(12) $$c^*(t) := I_1(t, Y(x) H(t)),$$

and there is a portfolio process $\pi^*(t)$, $t \in [0, T]$ with $(\pi^*, c^*) \in A'(x)$ and $X^{x,\pi^*,c^*}(T) = 0$ a.s..

b) The optimal terminal wealth B^* for the ***terminal wealth maximization problem***

$$\max_{(\pi, 0) \in A'(x)} E\left(U_2\left(X^{\pi}(T)\right)\right)$$

is given by

(13) $$B^* := I_2(Y(x) H(T)),$$

and there exists portfolio process $\pi^* \in A'(x)$ with $X^{x,\pi^*,c^*}(\mathrm{T}) = B^*$ a.s..

Step 2: ***Computation of the Optimal Strategy – the Representation Problem (R)***

An example: Log-utility and final wealth maximization, i.e. $U_2(x) = ln(x),\ \ U_1(t,x) = 0$ constant coefficients, d = 1.

$\Rightarrow$ Compute: $I_2(.),\ X(y),\ Y(x),....$

Use Corollary b)

$$\Rightarrow \qquad B^* := I_2(Y(x)H(T)) = x\frac{1}{H(T)} = xe^{\left(r+\frac{1}{2}\theta^2\right)T+\theta W(T)} = xe^{\left(\left(r+\theta^2\right)-\frac{1}{2}\theta^2\right)T+\theta W(T)}.$$

$\Rightarrow$ Guess the corresponding portfolio strategy from this explicit form as

$$\pi^*(t) = \frac{\theta}{\sigma} = \frac{b-r}{\sigma^2}. \qquad (14)$$

$\Rightarrow$ $B^* = X^{\pi^*}(T)$, i.e. we have $\pi^*(t) = \dfrac{b-r}{\sigma^2}$

Note: For arbitrary d we obtain $\boxed{\pi^*(t) = (\sigma\sigma')^{-1}(b - r\underline{1})}$

Method 1: “Comparison of coefficients”

Idea: Generalize the method of the example

- guess a process $X(t)$ with $X(0) = x$, $X(T) = B^*$ a.s.,
- write $X(t)$ as a functional of the underlying Brownian motion and the market coefficients,
- apply Itô's formula to this functional and compare drift and diffusion terms of the so obtained sde with those in the general form of the sde for a wealth process.

(see Theorem 2 below)

Method 2: "Feedback representation in the Markovian case"

More complicated, uses ideas of Malliavin calculus (see K. (1997) for an introduction)

Theorem 2
Assume the complete market setting of this section and that we have

(15) $$\frac{1}{H(t)}\,E\left(\int_t^T H(s)c^*(s)\,ds + H(T)B^* \middle| F_t\right) = f(t, W_1(t),\ldots, W_n(t))$$

for a non-negative function $f\in C^{1,2}([0,T)\times\mathbf{R}^n)$ with $f(0,\ldots,0) = x$. Then the optimal trading strategy $\varphi(t) = (\varphi_0(t),\ldots,\varphi_n(t))'$, $t\in[0,T]$, is given by

(16) $$\varphi_i(t) = \frac{1}{P_i(t)}\left(\sigma(t)^{-1}\nabla_x f(t,W_1(t),\ldots,W_n(t))\right)_i \text{ , } i = 1,\ldots, n,$$

(17) $$\varphi_0(t) = \left(X(t) - \sum_{i=1}^{n}\varphi_i(t)P_i(t)\right) / P_0(t) \text{ ,}$$

where $X(t)$ is the wealth process corresponding to the above trading strategy $\varphi(t)$ and the consumption process $c^*(t)$ of Theorem 1. $\nabla_X f(.)$ denotes the gradient of f with respect to the last n variables. The optimal portfolio process $\pi^*(t)$ of Theorem 1 is given by

(18) $$\pi^*(t) = \frac{1}{X(t)}\sigma(t)^{-1}\nabla_X f(t, W_1(t),\ldots, W_n(t)) \text{ .}$$

4. Optimal investment by stochastic control: The HJB-Equation

The classical continuous-time portfolio optimization approach by Merton (1969, 1971, …) does not use the completeness of the market.

Merton's idea:

Identify the portfolio optimization problem as a special case of a ***stochastic control problem***. Then, use standard results from stochastic control theory such as

- the Bellman principle
- the Hamilton-Jacobi-Bellman-Equation ("HJB-Equation")

Can be used as a cooking recipe, has a broader scope of application than the martingale, needs the complete solution of a non-linear partial differential equation …

$\Rightarrow$

We will give a short survey of stochastic control theory (see also Korn and Korn (2000))

Excursion: ***Solving stochastic control problems*** (for simplicity n=1)

Let $$v(t,x) := \max_{\pi(.)\big|_{[t,T]} \in A(x)} E^{t,x}\left(U\left(X^{\pi}(T)\right)\right)$$ ***value function***

Bellman-Principle:

$$v(t,x) = \max_{\pi(.)\big|_{[t,s]} \in A(x)} E^{t,x}\left(v\left(s, X^{\pi}(s)\right)\right)$$

Localize the BP: Apply the Itô-formula $\Rightarrow$

$$v(t,x) = v(t,x) + \\ + \max_{\pi(.)\big|_{[t,s]} \in A(x)} E^{t,x}\Bigg(\int_t^s \sigma\pi(u) X^{\pi}(u) v_x(.)\, dW(u) \\ + \int_t^s \left(v_t\left(u, X^{\pi}(u)\right) + \left(r + \pi(u)(b-r)\right) X^{\pi}(u) v_x(.) + \tfrac{1}{2}\sigma^2 \pi(u)^2 X^{\pi}(u) v_{xx}(.)\right) du\Bigg)$$

Divide by s–t, (formally) interchange the limit $s \downarrow t$ with the integration and expectation:

$\Rightarrow$

(19) $$\boxed{0 = \max_{\pi \in IR}\left(v_t(t,x) + \left(r + \pi(b-r)\right) x v_x(t.x) + \tfrac{1}{2}\sigma^2 \pi^2 x v_{xx}(t.x)\right)}$$

Theorem: ***Verification theorem for the solution of the Hamilton-Jacobi-Bellman-Equation***

If there exists a classical (i.e. a sufficiently differentiable) solution of the HJB-Equation

$$\sup_{\pi \in R}\left\{ v_t(t,x) + \left(r + \pi'(b-r)\right) x v_x(t,x) + \tfrac{1}{2}\pi'\sigma\sigma'\pi x^2 v_{xx}(t,x) \right\} = 0$$

$$v(T,x) = U(x)$$

that is polynomially bounded then we have $v(t,x) = \sup_{\pi(.)|_{[t,T]}} E^{t,x}\left(X^{\pi}(T)\right)$,

and an (admissible) portfolio process $\pi^*(t)$ (= $\pi^*(t, x)$) that yields the solution of the optimization in the HJB-Equation is an optimal portfolio process.

Algorithm for solving the portfolio problem

Step 1: Solve (formally) the optimization problem in the HJB-Equation
$\Rightarrow$ $\pi^*(t, x)$ (still depending on the unknown (!) value function and its derivatives).

Step 2: Insert π^* into the HJB-Equation, drop the sup-operator, solve the obtained partial differential equation explicitly.

Step 3: Check all the assumptions made during Steps 1 and 2 (very important, often forgotten !).

Example: HARA-utility function

Solve the problem

(20) $$\max_{\pi \in A'(x)} E^{0,x}\left(\frac{1}{\gamma}\left(X(T)\right)^{\gamma}\right), \quad \gamma < 1,\ \gamma \neq 0 \ \text{ fixed.}$$

with the value function

(21) $$v(t,x) = \max_{\pi \in A'(t,x)} E^{t,x}\left(\tfrac{1}{\gamma}\left(X(T)\right)^{\gamma}\right)$$

Corresponding HJB-Equation

(22) $$0 = \max_{\pi \in [a,b]^n} \left\{ \tfrac{1}{2}\pi'\sigma\sigma'\pi x^2 v_{xx}(t,x) + \left(\left(r + \pi'(b - r\underline{1})\right)x\right) v_x(t,x) + v_t(t,x) \right\}$$

(23) $$v(T,x) = \tfrac{1}{\gamma} x^{\gamma}$$

<u>**Step 1:**</u> "Solve the maximisation problem in the HJB-Equation"

(24) $$\pi^*(t,x) = -(\sigma\sigma')^{-1}(b - r\underline{1}) \frac{v_x(t,x)}{x v_{xx}(t,x)} \quad .$$

Important:

Note that we have implicitly assumed: $v_{xx} < 0,\ \pi^*(t,x) \in [a,b]^n, v \in C^{1,2}$**. (*)**

<u>Step 2:</u> "Solve the resulting partial differential equation"

Put $\pi^*(t,x)$ into equation (22), drop the sup-operator and obtain:

(24) $$0 = -\tfrac{1}{2}(b-r\underline{1})'(\sigma\sigma')^{-1}(b-r\underline{1})\frac{\left(v_x(t,x)\right)^2}{v_{xx}(t,x)} + rxv_x(t,x) + v_t(t,x), \quad v(T,x) = \tfrac{1}{\gamma}x^\gamma.$$

Ansatz:

(25) $$v(t,x) = \tfrac{1}{\gamma}x^\gamma f(t)$$ for some suitable function $f(t)$.

$\Rightarrow$

(26) $$f'(t) = -\left(r\gamma + \tfrac{1}{2}\tfrac{\gamma}{1-\gamma}(b-r\underline{1})'(\sigma\sigma')^{-1}(b-r\underline{1})\right)f(t), \quad f(T)=1$$

$\Rightarrow$

(27) $$f(t) = \exp\left(-\left(r\gamma + \tfrac{1}{2}\tfrac{\gamma}{1-\gamma}(b-r\underline{1})'(\sigma\sigma')^{-1}(b-r\underline{1})\right)(T-t)\right)$$

(28) $$\pi^*(t,x) = \frac{1}{1-\gamma}(\sigma\sigma')^{-1}(b-r\underline{1})$$

Step 3: "Check the assumptions"

$v(t,x) = \frac{1}{\gamma} x^{\gamma} f(t)$ according to (27) satisfies (*), $\pi^*(t,x)$ satisfies (*) for suitable constants a,b

Hence, choose them big enough and arrive at:

The optimal portfolio process is given by

(29) $\pi^*(t) = (\sigma\sigma')^{-1}(b - r\underline{1})\dfrac{1}{1-\gamma}$ **"Constant portfolio weigthts"**

Note the form of the optimal portfolio process and its dependence on γ, the risk aversion parameter!

Note also that the optimal wealth process has the form of

(30) $X(t) = x \cdot \exp\left(\left(r + \tfrac{1}{2}\tfrac{1}{1-\lambda}(b-r\underline{1})'(\sigma\sigma')^{-1}(b-r\underline{1}) - \pi^2\sigma^2\right)t + \tfrac{1}{1-\gamma}(b-r\underline{1})'(\sigma'\sigma)^{-1}\sigma W(t)\right)$

In particular, it is strictly positive !

More recipes for using the HJB-Equation technique:

i) Additional consumption

(31) $$v(t,x) := \sup_{(\pi,c)\in A'(t,x)} E\left(\int_0^T U_1(t,c(t))\,dt + U_2(X(T))\right)$$

with $A'(t,x)$ being the set of admissible strategies on [t,T] with initial wealth of x at time t.

$\Rightarrow$ Corresponding HJB-Equation

(32) $$0 = \sup_{c\geq 0,\pi\in[a,b]^n} \left\{ v_t(t,x) + \left(\left(r - \pi'(b - r\underline{1})\right)x - c\right)v_x(t,x) + \tfrac{1}{2}\pi'\sigma\sigma\pi x^2 v_{xx}(t,x) + U_1(t,c)\right\}$$

(33) $$v(T,x) = U_2(x)$$

ii) Finite time horizon with discounting

(34) $$v(t,x) = \sup_{(\pi,c)(t,x)} E^{t,x}\left(\int_t^T e^{-\rho(s-t)}U_1(c(s))\,ds + e^{-\rho(T-t)}U_2(X(T))\right).$$

$\Rightarrow$ Replace $U_1(t,c)$ in the HJB-Equation (32) by $U_1(c) - \rho v(t,x)$.

iii) Infinite time horizon with discounting

(35) $$v(x) = \sup_{(\pi,c)\in A'(x)} E^x\left(\int_0^\infty e^{-\rho s} U_1\big(c(s)\big)\,ds\right)$$

$\Rightarrow$

HJB-Equation related to this problem (no boundary condition !!!)

(36) $$0 = \sup_{(\pi,c)\in[a,b]^n\times[0,\infty)} \left\{\Big(\big(r+\pi'(b-r\underline{1})\big)x-c\Big)v_x(x)+\tfrac{1}{2}\pi'\sigma\sigma'\pi' x^2 v_{xx}(x)+U_1(c)-\rho v(x)\right\}$$

More on this: this afternoon …