Crash Hedging Strategies and Optimal Portfolios

by

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1 Introduction

1.1 Optimal Investment in the Black–Scholes Setting

Most basic setting:

$$
\begin{aligned}
dP_{0,0}(t) &= P_{0,0}(t)\, r_0\, dt\,, & P_{0,0}(0) &= 1\,, & &\text{“bond”}\\
dP_{0,1}(t) &= P_{0,1}(t)\left[\mu_0\, dt + \sigma_0\, dW_0(t)\right]\,, & P_{0,1}(0) &= p_1\,, & &\text{“stock”}
\end{aligned}
$$

with constant market coefficients μ_0, r_0, $\sigma_0 \neq 0$ and where W_0 is a Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$.

The **optimal portfolio problem** in this setting is to find a solution of

$$\boxed{\sup_{\boldsymbol{\pi}(\cdot)\in \boldsymbol{A}_0(\boldsymbol{x})} \mathbb{E}\left[\boldsymbol{U}\left(\boldsymbol{X}_0^{\boldsymbol{\pi}}(\boldsymbol{T})\right)\right],}$$

where U is the utility function of the investor, and X_0^π denotes the wealth process of the investor given the portfolio strategy π. More specific, the wealth process satisfies

$$\begin{aligned} dX_0^\pi(t) &= X_0^\pi(t)\left[(r_0 + \pi(t)[\mu_0 - r_0])\, dt + \pi(t)\sigma_0\, dW_0(t)\right], \\ X_0^\pi(0) &= x. \end{aligned}$$

Classical solution methods are

- the Martingale method (Pliska (1986), Karatzas et al. (1987), and Cox and Huang (1989)) and
- the stochastic control method (Merton (1969 and 1971)).

A solution of the optimal portfolio problem is called an **optimal portfolio strategy** and will be denoted by π^*. The most used utility functions and the corresponding optimal portfolio strategies are

i) **Logarithmic utility:** $U(x) = \ln(x)$ with $\pi_0^* = \frac{\mu_0 - r_0}{\sigma_0^2}$,

ii) **HARA–utility:** $U(x) = \frac{1}{\gamma}x^\gamma$ with $\gamma < 1$, $\gamma \neq 0$ and with

$$\pi_{0,\gamma}^* = \frac{1}{1-\gamma}\frac{\mu_0 - r_0}{\sigma_0^2},$$

iii) **Exponential utility:** $U(x) = -\exp(-\lambda x)$ with $\lambda > 0$ and with

$$\pi_{0,\lambda}^*(t) = \frac{1}{\lambda X^{\pi_{0,\lambda}^*}(t)} \cdot \frac{\mu_0 - r_0}{\sigma_0^2} \cdot \exp(-r_0 (T - t)).$$

1.2 Traditional Crash Modelling

Well–known empirical fact: Black–Scholes–price–model cannot explain large movements of real stock prices (the so–called "jumps" or "crashes").

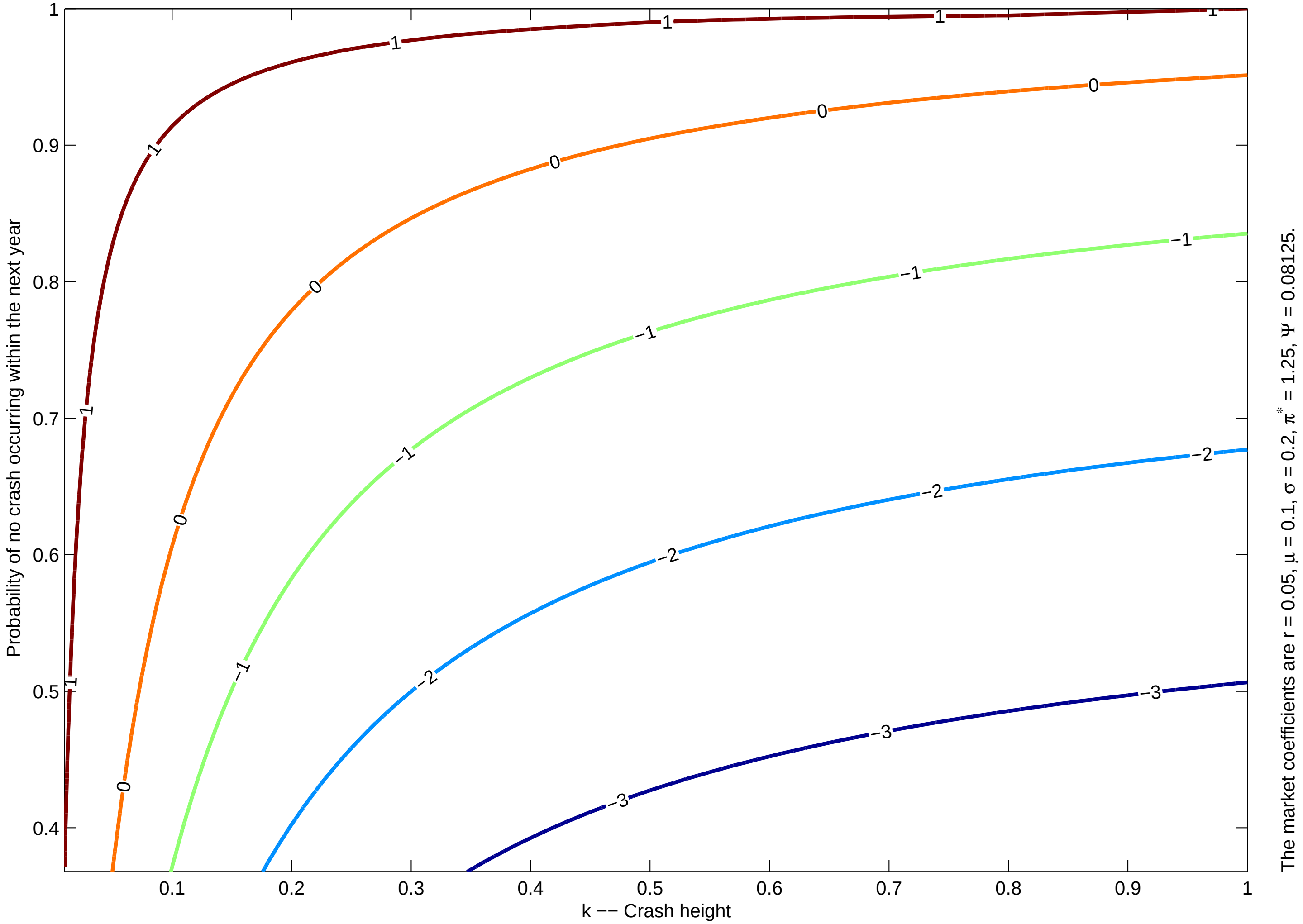
Examples: Merton (1976) or Aase (1984).

$$dP_{0,1}(t) = P_{0,1}(t)\left[\mu_0\, dt + \sigma_0\, dW_0(t) - k\, dN(t)\right], \qquad P_{0,1}(0) = p_1 .$$

where N is a Poisson process with intensity λ and $k > 0$ is the jump size. In the logarithmic–utility case $U(x) = \ln(x)$ the optimal portfolio strategy π_p^* calculates to

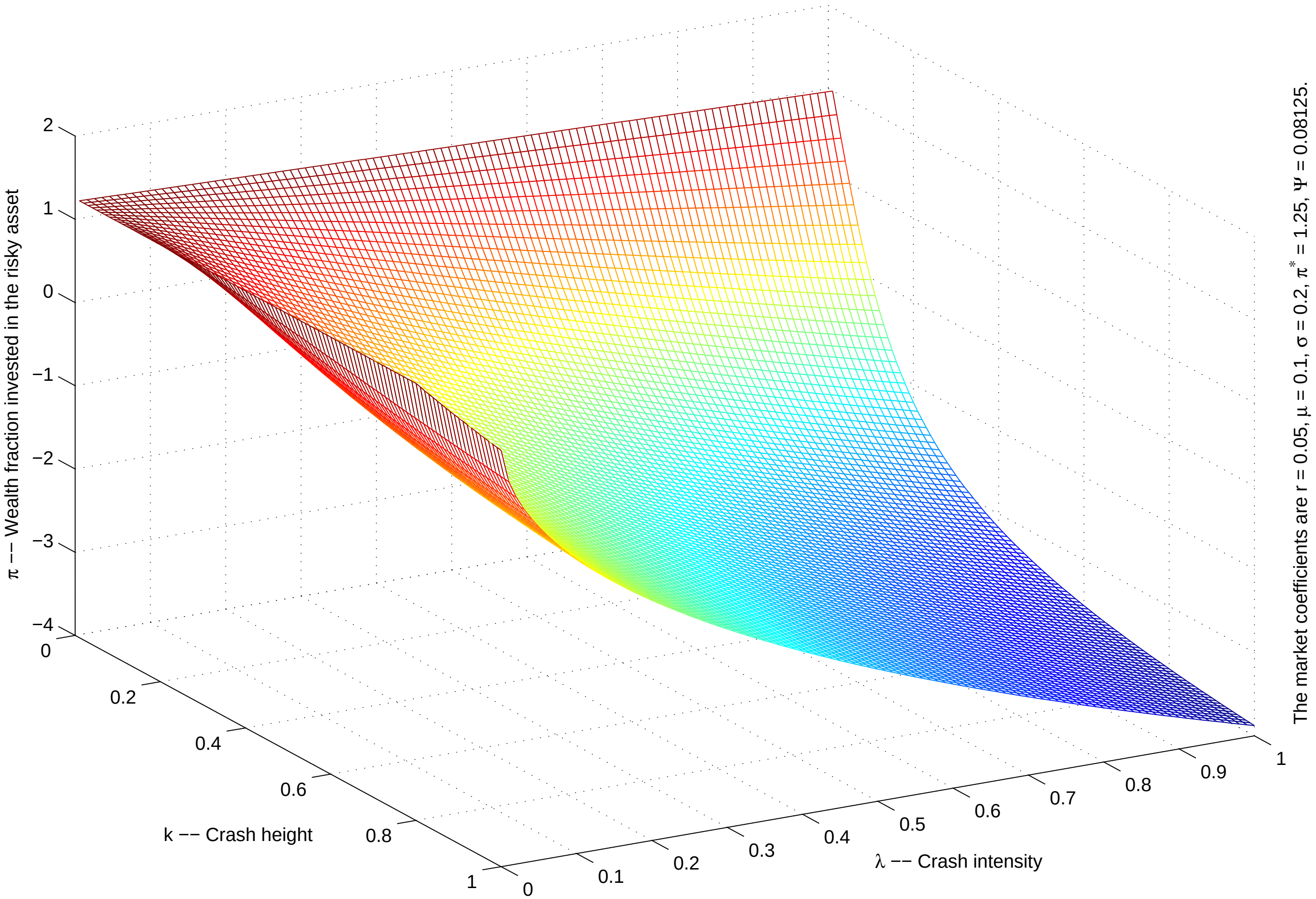
$$\pi_p^* = \frac{\sigma_0^2 + k\,(\mu_0 - r_0)}{2k\sigma_0^2} - \sqrt{\frac{k\lambda - (\mu_0 - r_0)}{k\sigma_0^2} + \left(\frac{\sigma_0^2 + k\,(\mu_0 - r_0)}{2k\sigma_0^2}\right)^2} .$$

Level Lines of the Optimal Portfolio Strategie π_P^*



The variables are the crash height k and the probability that no crash occurs within the next year. The market coefficients are assumed to be $r_0 = 0.05$, $\mu_0 = 0.1$ and $\sigma_0 = 0.2$.

The Dependence of the Optimal Portfolio Strategie π^*_P on the Crash Height and the Crash Intensity



This graphic shows the dependence of π^*_P from the crash height k and the crash intensity λ. The market coefficients are assumed to be $r_0 = 0.05$, $\mu_0 = 0.1$ and $\sigma_0 = 0.2$.

Remarks:

- Although these processes deliver a better fit, they **do not help** to determine optimal portfolio strategies under the threat of a crash!

- The **time to maturity** is very important in crash modelling. However, this variable is neglected in traditional crash modelling.

- **Asymmetry of risk** is not taken into consideration:
  - For $\pi \in [0, 1]$, you can loose at most your investment in the stock.
  - For $\pi < 0$ or $\pi > 1$, however, you can loose **much more** than just your investment in the stock!

1.3 Alternative Crash Modelling

1. Hua and Wilmott (1997) $\rightarrow$ Number and size of crashes in a given time interval are bounded. $\Rightarrow$ **No** probabilistic assumptions on height, number and times of occurence of crashes.

2. Korn and Wilmott (2002) $\rightarrow$ Determine worst case bounds for the performance of optimal investment.

For simplicity: One bond, one stock, at most one crash in $[0, T]$ with a crash height of k with $0 < k_* \leq k \leq k^* < 1$. Security prices in "normal times":

$$\begin{array}{rcllr} dP_{0,0}(t) & = & P_{0,0}(t)\, r_0\, dt\,, & P_{0,0}(0) = 1\,, & \text{"bond"} \\ dP_{0,1}(t) & = & P_{0,1}(t)\left[\mu_0\, dt + \sigma_0\, dW_0(t)\right]\,, & P_{0,1}(0) = p_1\,, & \text{"stock"} \end{array}$$

At crash time: stock price falls by a factor of $k \in [k_*, k^*]$.

Consequence: The wealth process $X_0^\pi(t)$ at crash time t satisfies:

$$\begin{array}{rcl} X_0^\pi(t-) & = & (1-\pi(t))\, X_0^\pi(t-) + \pi(t) X_0^\pi(t-) \\ \Longrightarrow & & (1-\pi(t))\, X_0^\pi(t-) + \pi(t) X_0^\pi(t-)\,(1-k) \\ & = & \boxed{\boldsymbol{(1-\pi(t)k) \cdot X_0^\pi(t-) = X_0^\pi(t)}.} \end{array}$$

Thus: Following the portfolio process $\pi(\cdot)$ if a crash of size k happens at time t leads to a final wealth of

$$X^{\pi}(T) = (1 - \pi(t)k) \cdot X_0^{\pi}(T),$$

if $X_0^{\pi}(\cdot)$ denotes the wealth process in the model without any crash.

Hence:

- "High" values of $\pi(\cdot)$ lead to a high final wealth if no crash occurs at all, but to a high loss at the crash time.
- "Low" values of $\pi(\cdot)$ lead to a low final wealth if no crash occurs at all, but to a small loss (or even no loss at all!) at the crash time.

Moral: We have two competing aspects ("Hedging vs. Return") for two different scenarios ("Crash or not") and are therefore faced with a **balanced problem between crash impact minimization and return maximization.**

Aim: Find the best uniform **worst case bound**, e.g. solve

$$\boxed{\sup_{\boldsymbol{\pi}(\cdot)\in \boldsymbol{A}_0(\boldsymbol{x})} \inf_{\substack{0\le\boldsymbol{\tau}\le \boldsymbol{T}\\ \boldsymbol{k}\in\boldsymbol{K}}} \mathbb{E}\left[\boldsymbol{U}\left(\boldsymbol{X}^{\boldsymbol{\pi}}\left(\boldsymbol{T}\right)\right)\right],}$$

where the final wealth satisfies $X^{\pi}(T) = (1-\pi(\tau)k)\, X_0^{\pi}(T)$ in the case of a crash of size k at stopping time τ. Moreover, $K = \{0\} \cup [k_*, k^*]$.

Note: To avoid bankruptcy we require $\pi(t) < \frac{1}{k^*}$ for all $t \in [0, T]$.

Important Remarks: (!)

- We do **not (!)** compare two different strategies **scenario-wise.** Typically, two different strategies have two different worst case scenarios!

- The worst case bounds do **not** depend on the probability of the worst case!

- Assuming $\mu_0 > r_0$, we do not have to consider portfolio processes $\pi(t)$ that can attain negative values since the utility function is increasing in x.

Two extreme strategies (in the logarithmic utility case):

1. **"Playing safe":**
   $\pi(t) \equiv 0 \Longrightarrow$ worst case scenario: no crash (!), leading to the following worst case bound of

$$WCB_0 = \mathbb{E}\left[\ln\left(X^0(T)\right)\right] = \ln(x) + r_0 T.$$

2. **"Optimal investment in the crash–free world":**
   $\pi(t) \equiv \pi_0^* = \frac{\mu_0 - r_0}{\sigma_0^2} \Longrightarrow$ worst case scenario: a crash of maximum size k^* (at any arbitrary time (!)), leading to the following worst case bound of

$$\begin{aligned} WCB_{\pi_0^*} = \mathbb{E}\left[\ln\left(X^{\pi_0^*}(T)\right)\right] &= \ln(x) + r_0 T + \frac{1}{2}\left(\frac{\mu_0 - r_0}{\sigma_0}\right)^2 T \\ &\quad + \ln\left(1 - \pi_0^* k^*\right). \end{aligned}$$

Insights:

- it depends on the time to maturity which one of the above strategies is better.

- strategy 1 takes too few risk to be good if no crash occurs while strategy 2 is too risky to perform well if a crash occurs $\Longrightarrow$ the optimal strategy should balance this out!

- a constant portfolio process **cannot** be the optimal one.

2 Optimal Investment under the Threat of a Crash

This chapter is based on Korn and Wilmott (2002), Korn and M. (2005), and M. (2006).

2.1 The Set up

$\boldsymbol{U(x) = \ln(x)}$, the price of the bond and the risky asset are assumed to be given by

$$
\begin{aligned}
dP_{1,0}(t) &= P_{1,0}(t)\, r_1\, dt\,, & P_{1,0}(\tau) = P_{0,0}(\tau)\,,\\
dP_{1,1}(t) &= P_{1,1}(t)\left[\mu_1\, dt + \sigma_1\, dW_1(t)\right]\,, & P_{1,1}(\tau) = (1-k)\, P_{0,1}(\tau)\,,
\end{aligned}
$$

respectively, with constant market coefficients r_1, μ_1 and $\sigma_1 \neq 0$ **after a possible crash** of size k at time τ.

For simplicity, the initial market will also be called market 0, while the market after a crash will be called market 1.

Definition 2.1

1. *The* **wealth process** $X^\pi(t)$ *in the crash model is defined as*

$$X^\pi(t) = \left\{ \begin{array}{ll} X_0^\pi(t) & \text{for } 0 \le t < \tau \\ [1-\pi(\tau)k]\, X_1^{\pi,\tau,X_0^\pi(\tau)}(t) & \text{for } t \ge \tau \ge 0\,, \end{array} \right\} \tag{1}$$

   given the occurrence of a jump of height k at time τ, is strictly positive.

2. *The problem to solve*

$$\sup_{\pi(\cdot)\in A(x)} \inf_{\substack{0\le\tau\le T,\\ k\in K}} \mathbb{E}\left[\ln\left(X^\pi(T)\right)\right]\,, \tag{2}$$

   where the final wealth $X^\pi(T)$ in the case of a crash of size k at time τ is given by

$$X^\pi(T) = [1-\pi(\tau)k]\, X_1^{\pi,\tau,X_0^\pi(\tau)}(T)\,, \tag{3}$$

   with $X_1^{\pi,\tau,X_0^\pi(\tau)}(t)$ as above, is called the **worst case scenario portfolio problem**.

Definition 2.2

1. *The* **value function** *to the above problem is defined via*

$$\nu_c(t,x) = \sup_{\pi(\cdot)\in A(t,x)} \inf_{\substack{t\leq\tau\leq T,\\ k\in K}} \mathbb{E}\left[\ln\left(X^{\pi,t,x}(T)\right)\right] . \tag{4}$$

2. *The* **value function in the crash–free setting** *of market i will be denoted by*

$$\nu_i(t,x) = \sup_{\pi(\cdot)\in A_i(t,x)} \mathbb{E}\left[\ln\left(X_i^{\pi,t,x}(T)\right)\right]$$

for $i = 0, 1$.

Definition 2.3
For $i = 0, 1$ let us name

1. *the* **optimal portfolio strategy** *in market i, assuming that no crash will happen, by*

$$\pi_i^* := \frac{\mu_i - r_i}{\sigma_i^2}.$$

2. *Moreover,*

$$\Psi_i := r_i + \frac{1}{2}\left(\frac{\mu_i - r_i}{\sigma_i}\right)^2 = r_i + \frac{\sigma_i^2}{2}\left(\pi_i^*\right)^2$$

will be called the **utility growth potential** *or* **earning potential** *of market i.*

Define for an arbitrary admissible portfolio strategy $\pi(t)$

$$\begin{aligned}
\nu_\pi(t,x) &:= \mathbb{E}\left[\ln\left(X_0^{\pi,t,x}(T)\right)\right] \\
&= \ln(x) + \mathbb{E}\left[\int_t^T \left[\pi(s)(\mu_0 - r_0) + r_0 - \frac{1}{2}\pi^2(s)\sigma_0^2\right] ds\right] \\
&= \ln(x) - \frac{\sigma_0^2}{2}\mathbb{E}\left[\int_t^T \left[(\pi(s) - \pi_0^*)^2 - \frac{2}{\sigma_0^2}\Psi_0\right] ds\right] \\
&= \ln(x) + \mathbb{E}\left[\int_t^T \left[\Psi_0 - \frac{\sigma_0^2}{2}(\pi(s) - \pi_0^*)^2\right] ds\right].
\end{aligned}$$

In particular, the value function of market i given that no crash occurs is

$$\nu_i(t,x) = \ln(x) + \Psi_i(T-t).$$

2.2 A Main Result

Definition 2.4

1. *A portfolio strategy $\widehat{\pi}$ determined via the equation*

$$\widehat{\nu}\,(t,x) = \left\{ \begin{array}{ll} \nu_1\,(t, x\,(1-\widehat{\pi}(t)k^*)) & \text{for } \widehat{\pi}(t) \geq 0 \\ \nu_1\,(t, x\,(1-\widehat{\pi}(t)k_*)) & \text{for } \widehat{\pi}(t) < 0 \end{array} \right\} \text{ for all } t \in [0,T]$$

   will be called a **crash hedging strategy**.

2. *A portfolio strategy $\tilde{\pi}$ is a* **partial crash hedging strategy**, *if there exists an $S \in (0,T)$ such that $\tilde{\pi}$ is a crash hedging strategy on $[0,S]$ and is a solution to the worst case scenario portfolio problem on $[S,T]$.*

Hereby, the convention $\widehat{\nu}(t,x) := \nu_{\widehat{\pi}}(t,x)$ is used.

Rewriting the determining equation for the non–negative crash hedging strategy $\widehat{\pi}$ gives

$$
\begin{aligned}
\widehat{\nu}\,(t,x) &= \nu_1\left(t, x\left(1-\widehat{\pi}(t)k^*\right)\right) \\
\Longleftrightarrow \quad \ln(x) + \int_t^T \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\widehat{\pi}(s)-\pi_0^*\right)^2\right] ds & \\
&= \ln(x) + \ln\left(1-\widehat{\pi}(t)k^*\right) + \Psi_1\,(T-t) \\
\Longleftrightarrow \quad \ln\left(1-\widehat{\pi}(t)k^*\right) &= \int_t^T \left[\Psi_0 - \Psi_1 - \frac{\sigma_0^2}{2}\left(\widehat{\pi}(s)-\pi_0^*\right)^2\right] ds\,. \quad (5)
\end{aligned}
$$

Assuming that $\widehat{\pi}$ is differentiable, differentiating with respect to t yields

$$
\begin{aligned}
\frac{-\widehat{\pi}'(t)k^*}{1-\widehat{\pi}(t)k^*} &= \frac{\sigma_0^2}{2}\left(\widehat{\pi}(t)-\pi_0^*\right)^2 + \Psi_1 - \Psi_0 \\
\Longleftrightarrow \quad \widehat{\pi}'(t) &= \left(\widehat{\pi}(t) - \frac{1}{k^*}\right)\left[\frac{\sigma_0^2}{2}\left(\widehat{\pi}(t)-\pi_0^*\right)^2 + \Psi_1 - \Psi_0\right].
\end{aligned}
$$

Theorem 2.5

1. *If $\Psi_1 \geq r_0$, then there exists a unique crash hedging strategy $\widehat{\pi}$, which is given by the solution of the differential equation*

$$\widehat{\pi}'(t) = \left(\widehat{\pi}(t) - \frac{1}{k^*}\right)\left[\frac{\sigma_0^2}{2}\left(\widehat{\pi}(t) - \pi_0^*\right)^2 + \Psi_1 - \Psi_0\right], \quad (6)$$

$$\text{and} \quad \widehat{\pi}(T) = 0\,. \quad (7)$$

Moreover, this crash hedging strategy is bounded by $0 \leq \widehat{\pi} \leq \frac{1}{k^*}$, if $\Psi_1 > \Psi_0$. In the case of $\Psi_1 \leq \Psi_0$, the crash hedging strategy is bounded by $0 \leq \widehat{\pi} \leq \pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)}$.

2. *If $\Psi_1 < r_0$, then there exists a unique crash hedging strategy $\widehat{\pi}$, which is given by the solution of the differential equation*

$$\widehat{\pi}'(t) = \left(\widehat{\pi}(t) - \frac{1}{k_*}\right)\left[\frac{\sigma_0^2}{2}\left(\widehat{\pi}(t) - \pi_0^*\right)^2 + \Psi_1 - \Psi_0\right], \quad (8)$$

$$\text{and} \quad \widehat{\pi}(T) = 0\,. \quad (9)$$

Furthermore, this crash hedging strategy is bounded by $\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)} \leq \widehat{\pi}(t) < 0$ for $t \in [0, T)$.

3. *If $\Psi_1 < \Psi_0$ and $\pi_0^* < 0$, there exists a partial crash hedging strategy $\tilde{\pi}$ (which is different from $\widehat{\pi}$), if*

$$S := T - \frac{\ln\left(1 - \pi_0^* k_*\right)}{\Psi_0 - \Psi_1} > 0\,. \tag{10}$$

With this, $\tilde{\pi}$ is on $[0, S]$ given by the unique solution of the differential equation

$$\begin{aligned} \tilde{\pi}'(t) &= \left(\tilde{\pi}(t) - \frac{1}{k_*}\right)\left[\frac{\sigma_0^2}{2}\left(\tilde{\pi}(t) - \pi_0^*\right)^2 + \Psi_1 - \Psi_0\right], & (11)\\ \text{and} \quad \tilde{\pi}(S) &= \pi_0^*\,. & (12) \end{aligned}$$

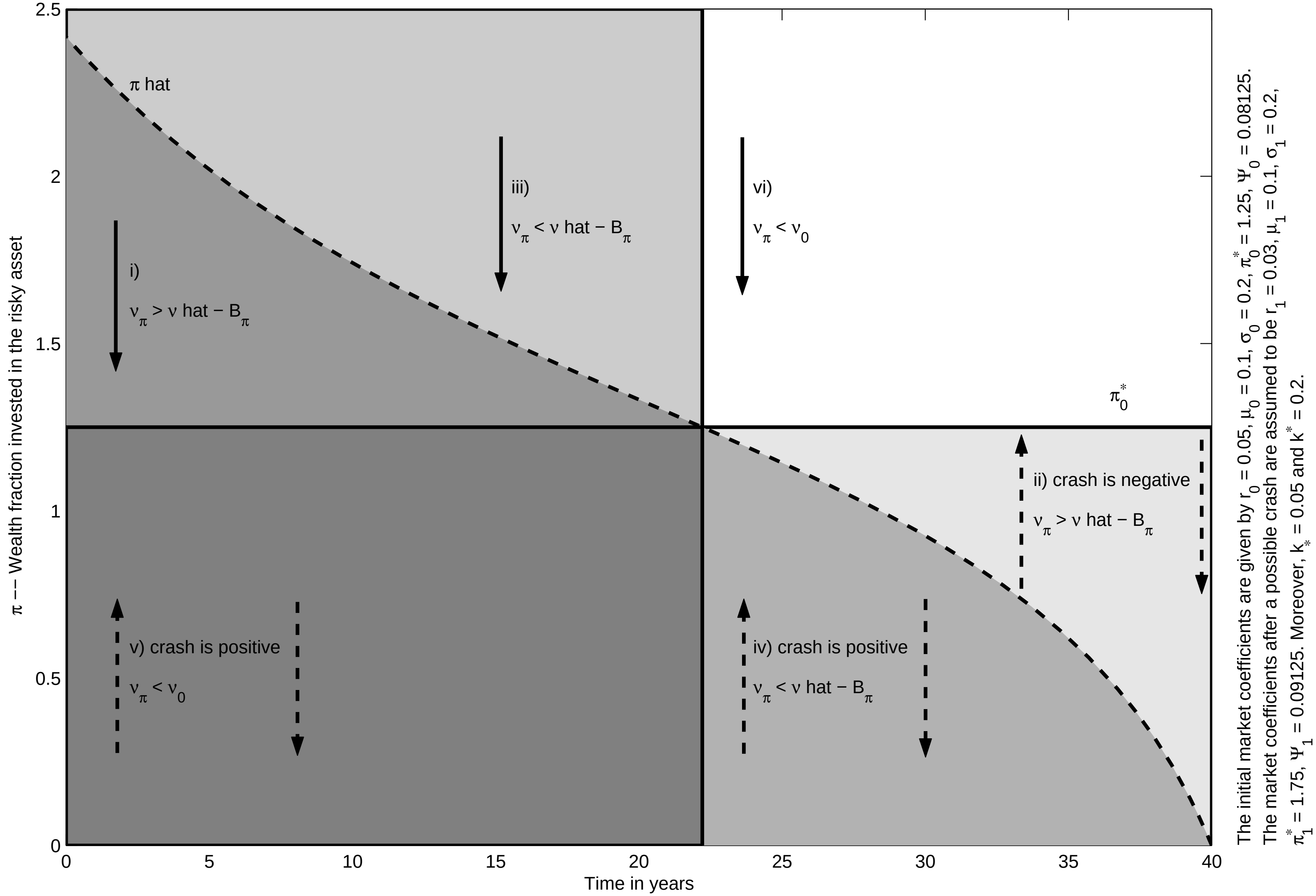
On $[S, T]$ set $\tilde{\pi}(t) := \pi_0^*$. This partial crash hedging strategy is bounded by $\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)} \leq \tilde{\pi} \leq \pi_0^* < 0$.

The optimal portfolio strategy for an investor, who wants to maximize her worst case scenario portfolio problem, is given by

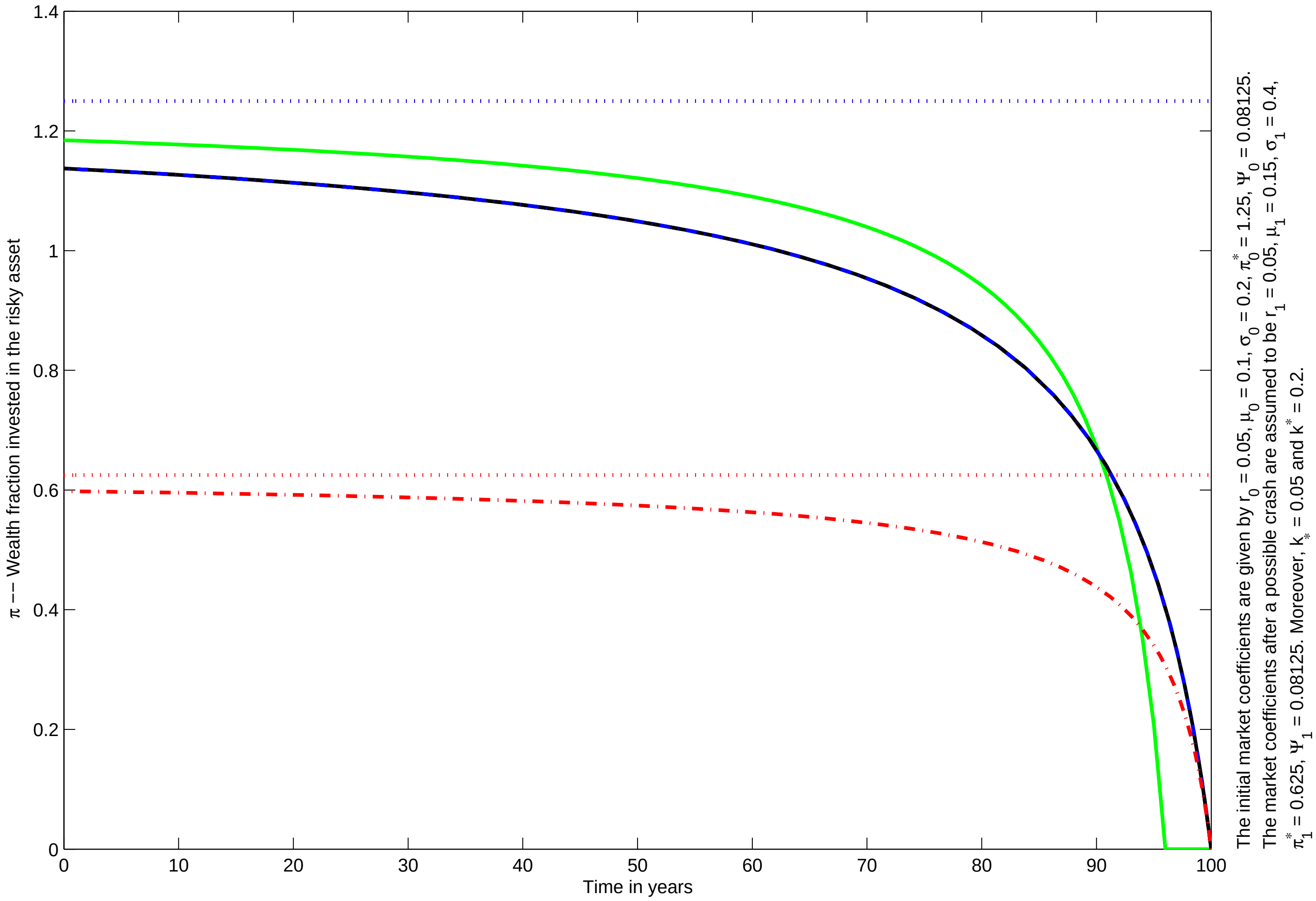
$$\bar{\pi}(t) := \min\left\{\widehat{\pi}(t), \tilde{\pi}(t), \pi_0^*\right\} \qquad \text{for all } t \in [0, T], \tag{13}$$

where $\tilde{\pi}(t)$ is only taken into account if it exists. $\bar{\pi}$ will be named the **optimal crash hedging strategy**.

Geometric Interpretation of the Crash Hedging Strategy

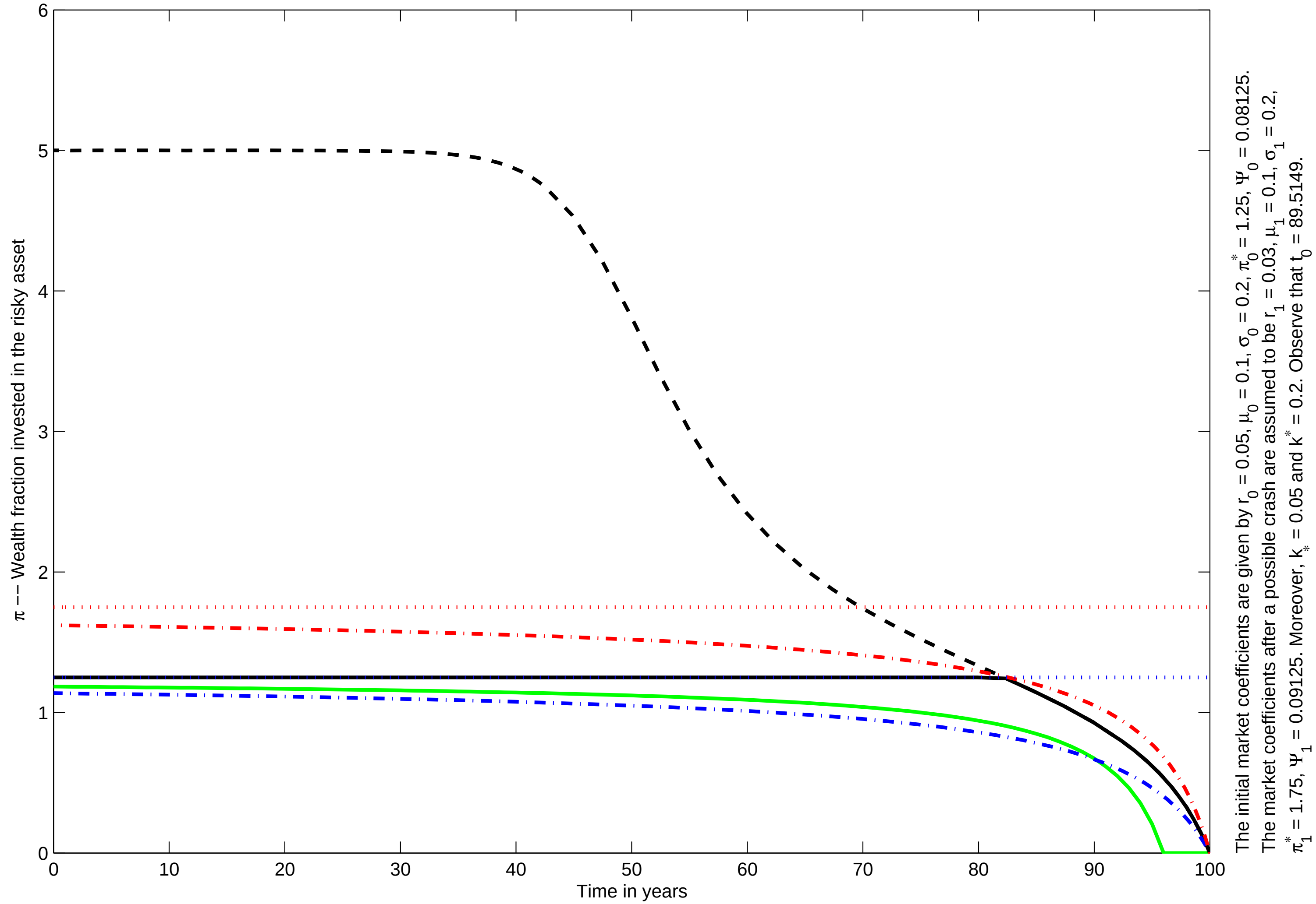


Example for $\Psi_1 = \Psi_0$ and $\pi_0^* \geq 0$



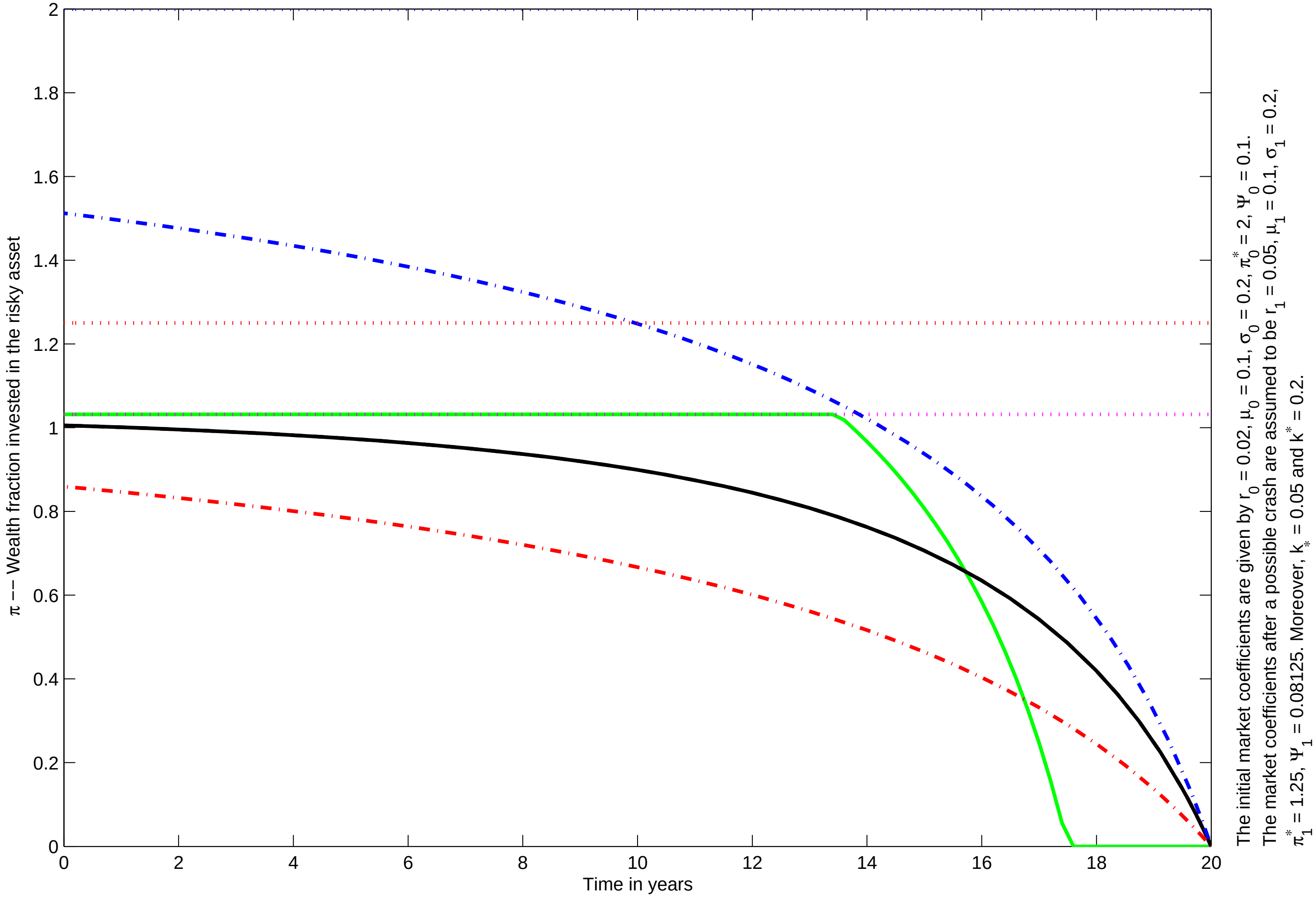
This graphic shows $\widehat{\pi} = \bar{\pi} = \widehat{\phi}_0$ (blue dash–dotted line with black background), $\widehat{\varphi} = \pi_0^*$ (blue dotted line), $\bar{\varphi}$ (green line), $\widehat{\phi}_1$ (red dash–dotted line), and π_1^* (red dotted line).

Example for $\Psi_1 > \Psi_0$ and $\pi_0^* \geq 0$



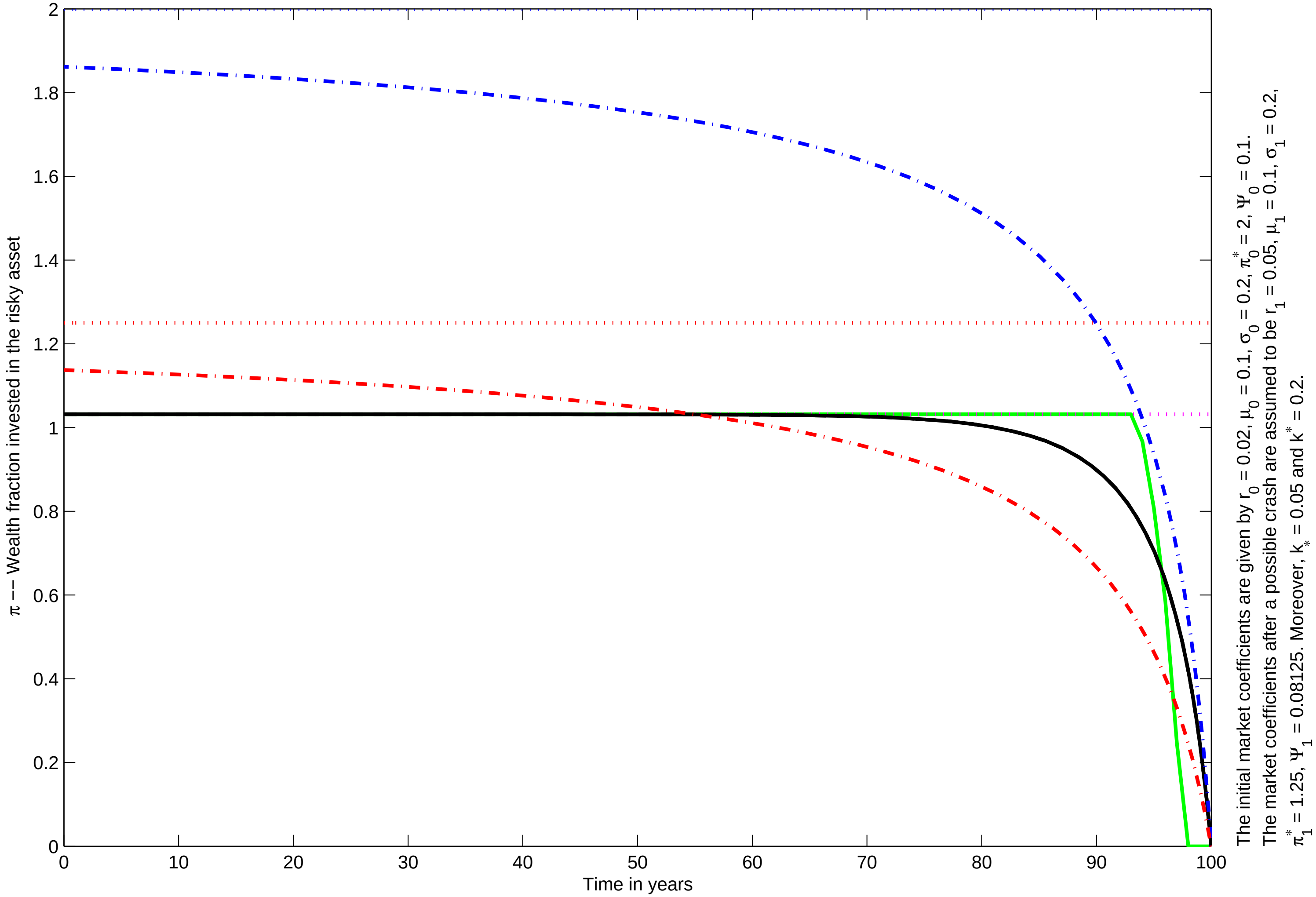
This graphic shows $\widehat{\pi}$ (black dashed line), $\bar{\pi}$ (black line), $\bar{\varphi}$ (green line), $\widehat{\phi}_0$ (blue dash–dotted line), $\widehat{\phi}_1$ (red dash–dotted line), π_0^* (blue dotted line), and π_1^* (red dotted line).

Example for $r_0 \leq \Psi_1 \leq \Psi_0$ and $\pi_0^* \geq 0$



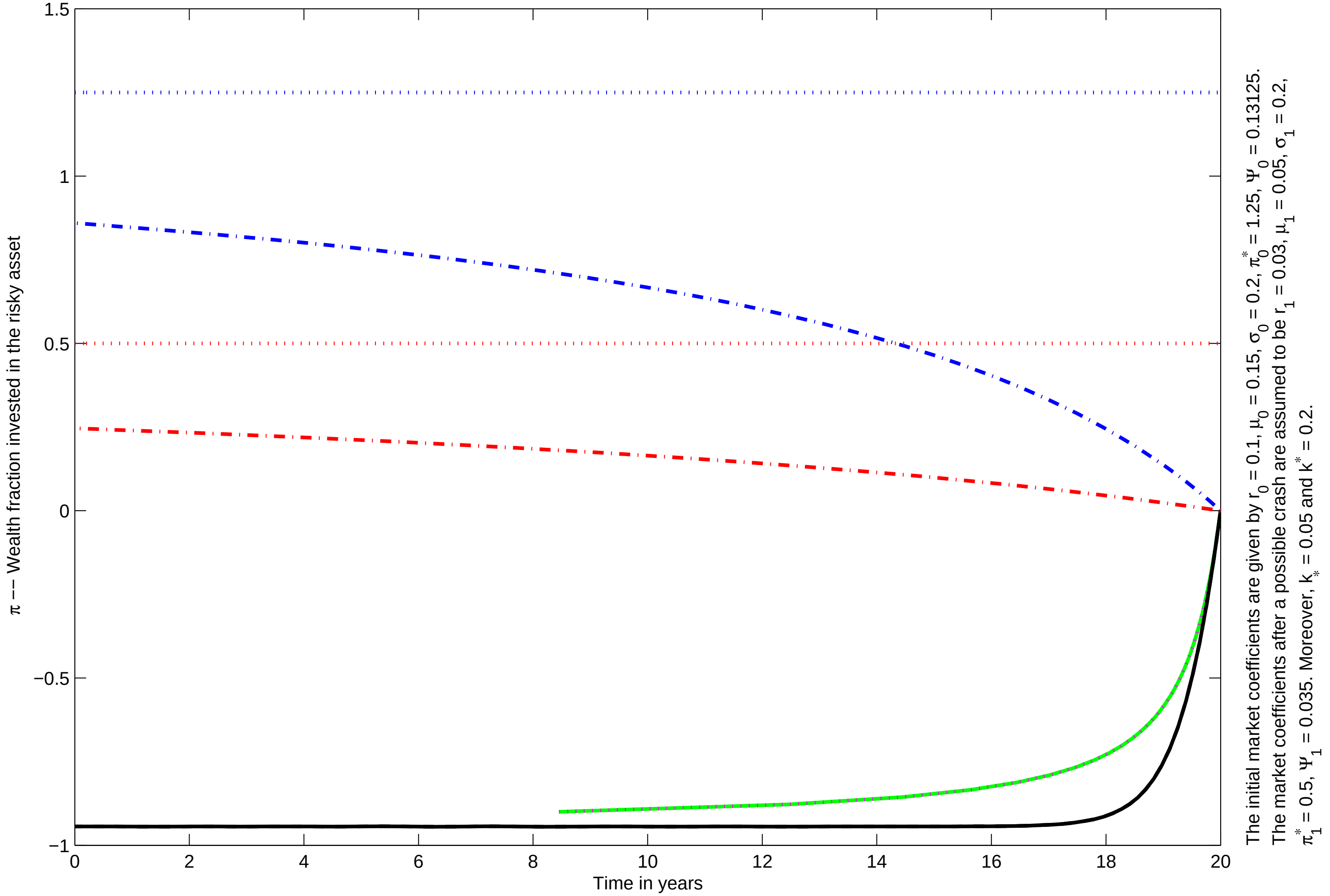
This graphic shows $\widehat{\pi} = \bar{\pi}$ (black line), $\widehat{\varphi}$ (cyan dotted line), $\bar{\varphi}$ (green line), $\widehat{\phi}_0$ (blue dash–dotted line), $\widehat{\phi}_1$ (red dash–dotted line), $\pi_0^* = 2$ (blue dotted line), and π_1^* (red dotted line).

Example for $r_0 \leq \Psi_1 \leq \Psi_0$ and $\pi_0^* \geq 0$, the long term behaviour



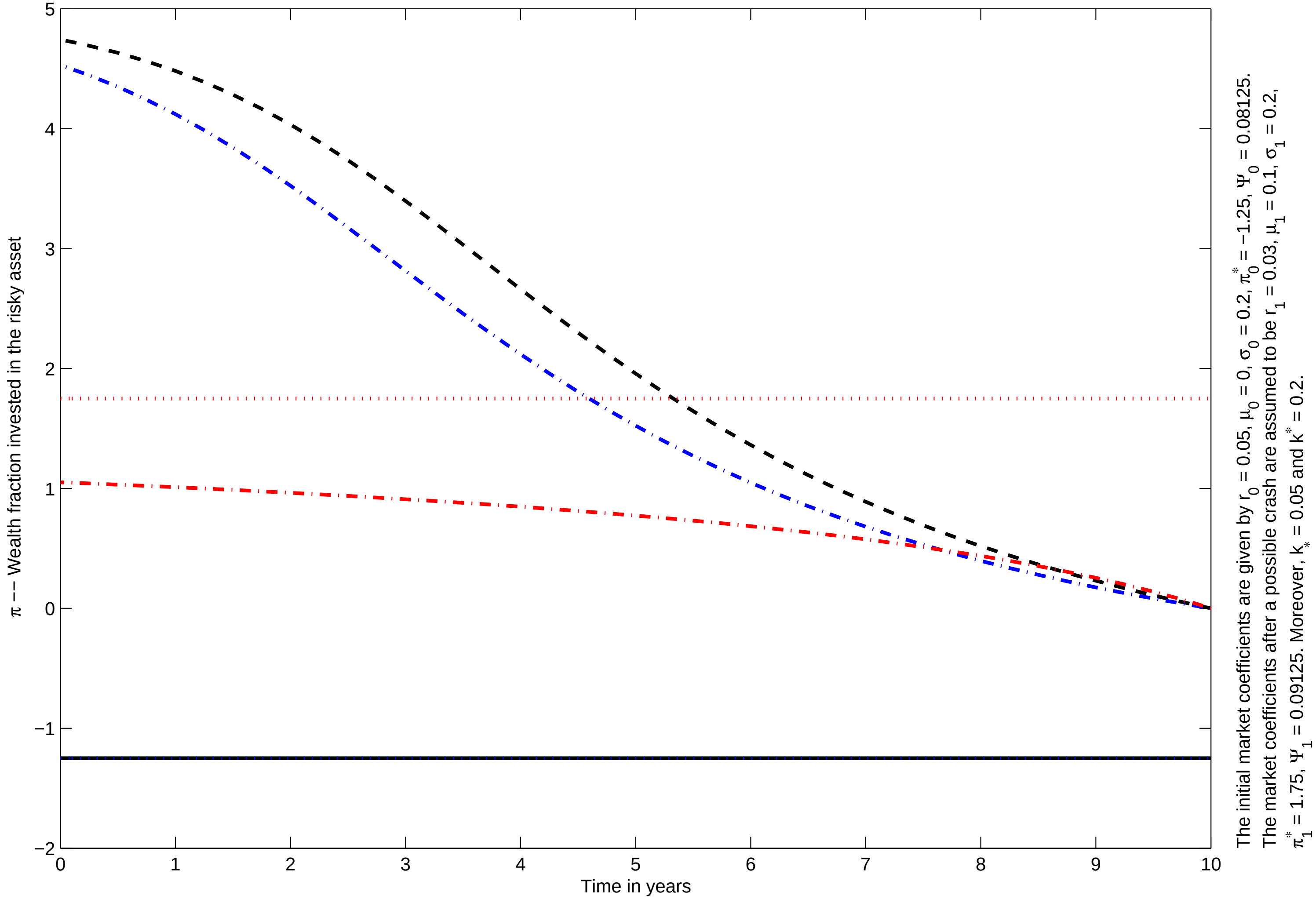
This graphic shows the long term behaviour of $\widehat{\pi} = \bar{\pi}$ (black line), $\widehat{\varphi}$ (cyan dotted line), $\bar{\varphi}$ (green line), $\widehat{\phi}_0$ (blue dash–dotted line), $\widehat{\phi}_1$ (red dash–dotted line), $\pi_0^* = 2$ (blue dotted line), and π_1^* (red dotted line).

Example for $\Psi_1 < r_0$ and $\pi_0^* \geq 0$



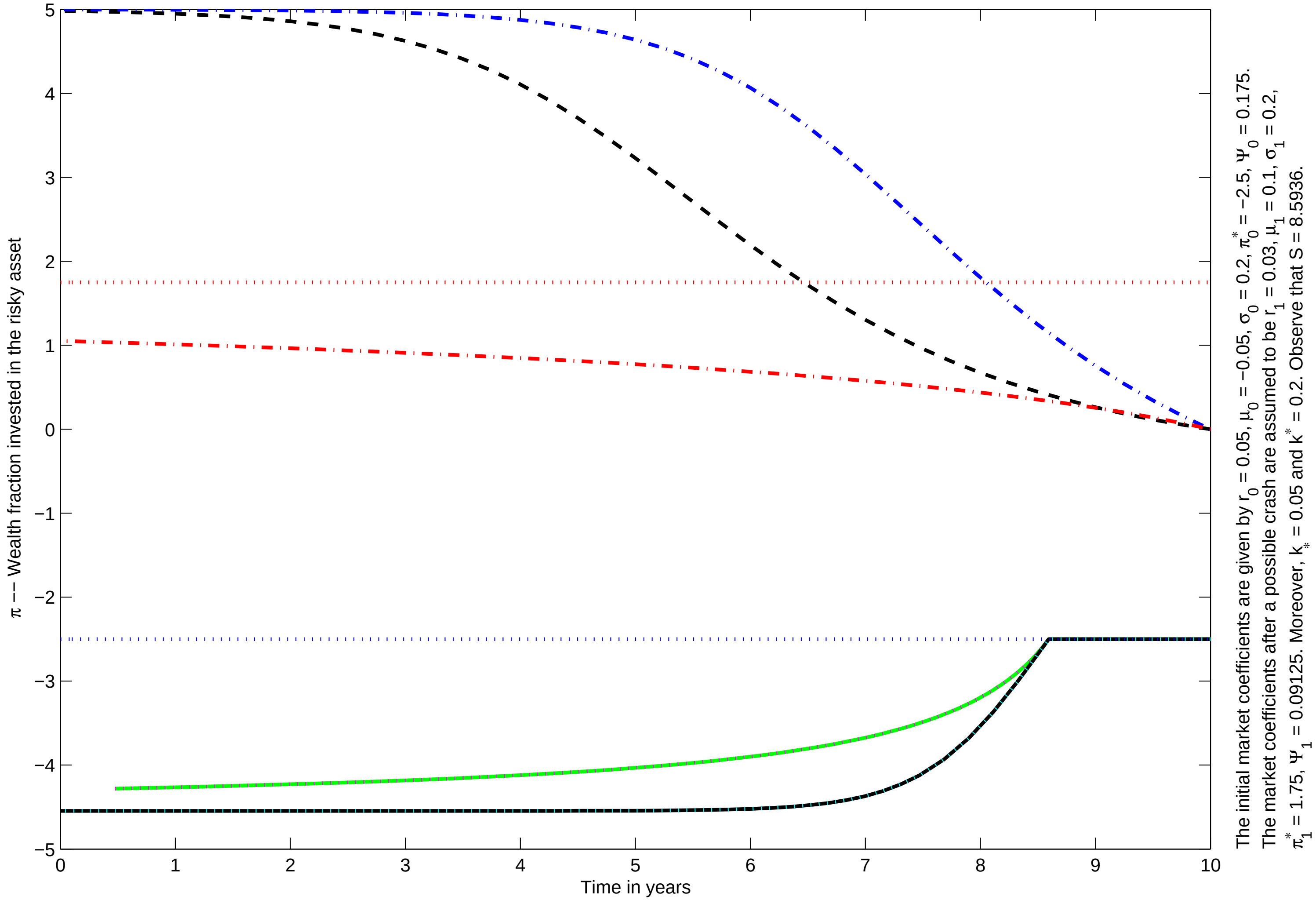
This graphic shows $\widehat{\pi} = \overline{\pi}$ (black line), $\bar{\varphi} = \widehat{\varphi}$ (green line with cyan dotted points), $\widehat{\phi}_0$ (blue dash–dotted line), $\widehat{\phi}_1$ (red dash–dotted line), π_0^* (blue dotted line), and π_1^* (red dotted line).

Example for $\Psi_1 > \Psi_0$ and $\pi_0^* < 0$



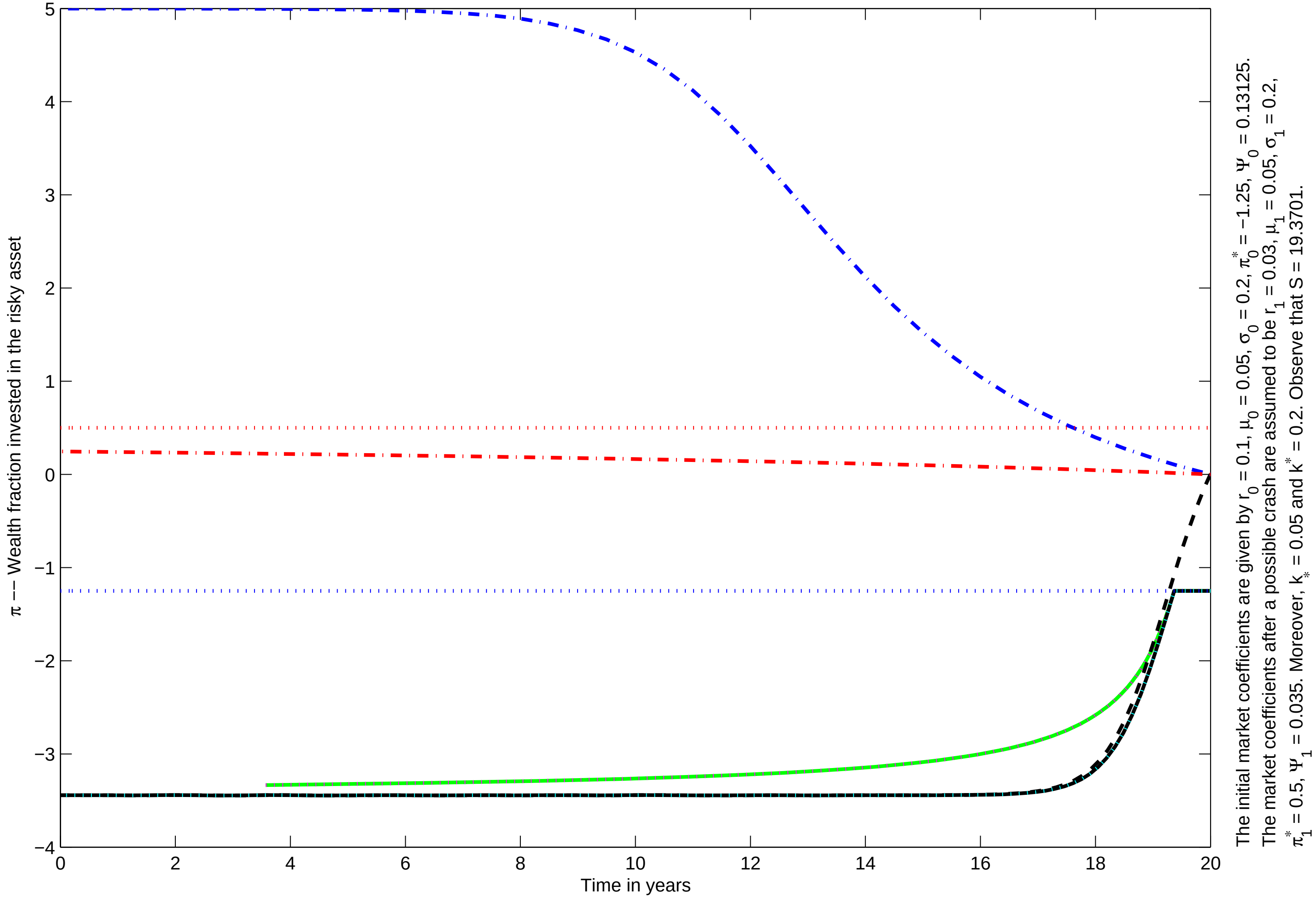
This graphic shows $\widehat{\pi}$ (black dashed line), $\bar{\pi} = \bar{\varphi} = \pi_0^*$ (blue dotted line with black background), $\widehat{\phi}_0$ (blue dash–dotted line), $\widehat{\phi}_1$ (red dash–dotted line), and π_1^* (red dotted line).

Example for $r_0 \leq \Psi_1 \leq \Psi_0$ and $\pi_0^* < 0$



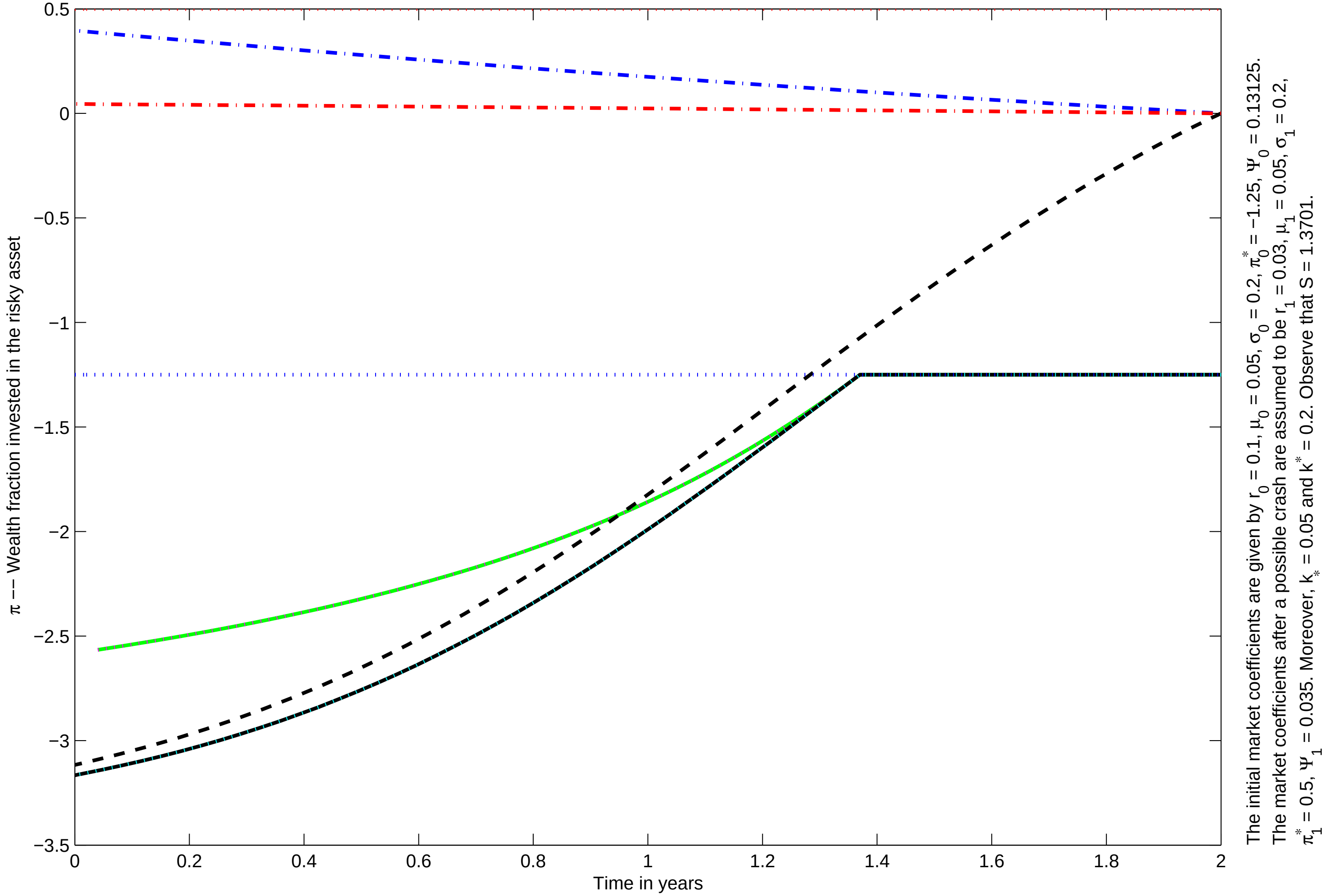
This graphic shows $\widehat{\pi}$ (black dashed line), $\bar{\pi} = \tilde{\pi}$ (black line), $\widehat{\varphi}$ (cyan dotted line), $\bar{\varphi}$ (green line), $\widehat{\phi}_0$ (blue dash–dotted line), $\widehat{\phi}_1$ (red dash–dotted line), π_0^* (blue dotted line), and π_1^* (red dotted line).

Example for $\Psi_1 < r_0$ and $\pi_0^* < 0$, the long term behaviour



This graphic shows $\hat{\pi}$ (black dashed line), $\bar{\pi} = \tilde{\pi}$ (black line), $\hat{\varphi}$ (cyan dotted line), $\bar{\varphi}$ (green line), $\widehat{\phi}_0$ (blue dash–dotted line), $\widehat{\phi}_1$ (red dash–dotted line), π_0^* (blue dotted line), and π_1^* (red dotted line).

Example for $\Psi_1 < r_0$ and $\pi_0^* < 0$



This graphic shows $\hat{\pi}$ (black dashed line), $\bar{\pi} = \tilde{\pi}$ (black line), $\hat{\varphi}$ (cyan dotted line), $\bar{\varphi}$ (green line), $\widehat{\phi}_0$ (blue dash–dotted line), $\widehat{\phi}_1$ (red dash–dotted line), π_0^* (blue dotted line), and π_1^* (red dotted line).

3.2 Optimal Portfolios Given the Probability of a Crash

In this section, let us suppose that the investor knows the probability of a crash occurring. Let p, with $p \in [0,1]$, be the probability of a crash happening. In this situation, the optimization problem writes to

$$\begin{aligned}
&\sup_{\pi(\cdot)\in A(t,x)} \inf_{\substack{t\le\tau\le T,\\ k\in K}} \mathbb{E}_p\left[\ln\left(X^{\pi,t,x}(T)\right)\right] \\
&:= \sup_{\pi(\cdot)\in A(t,x)} \left\{ p \cdot \left\{ \inf_{\substack{t\le\tau\le T,\\ k\in K}} \mathbb{E}\left[\ln\left(X^{\pi,t,x}(T)\right)\right] \right\} + (1-p)\,\mathbb{E}\left[\ln\left(X_0^{\pi,t,x}(T)\right)\right] \right\} \\
&= \sup_{\pi(\cdot)\in A(t,x)} \left\{ p \cdot \left\{ \inf_{\substack{t\le\tau\le T,\\ k\in K}} \mathbb{E}\left[\nu_1\left(\tau, X_0^{\pi,t,x}(\tau)\left(1-\pi(\tau)k\right)\right)\right] \right\} + (1-p)\,\mathbb{E}\left[\nu_\pi\left(t,x\right)\right] \right\}.
\end{aligned}$$

Observe that the two extremes, $p \in \{0,1\}$ are straightforward to solve:

- $\boldsymbol{p = 1}$: $$\sup_{\pi(\cdot)\in A(t,x)} \inf_{\substack{t\le\tau\le T,\\ k\in K}} \mathbb{E}_1\left[\ln\left(X^{\pi,t,x}(T)\right)\right] = \sup_{\pi(\cdot)\in A(t,x)} \inf_{\substack{t\le\tau\le T,\\ k\in K}} \mathbb{E}\left[\ln\left(X^{\pi,t,x}(T)\right)\right].$$

  Thus, this is the original worst case scenario portfolio problem. The solution is already known.

- $\boldsymbol{p = 0}$: $$\sup_{\pi(\cdot)\in A(t,x)} \inf_{\substack{t\le\tau\le T,\\ k\in K}} \mathbb{E}_0\left[\ln\left(X^{\pi,t,x}(T)\right)\right] = \sup_{\pi(\cdot)\in A(t,x)} \mathbb{E}\left[\ln\left(X_0^{\pi,t,x}(T)\right)\right],$$

  which is the classical optimal portfolio problem. The solution is well–known and is given in our notation (see Definition 1) by π_0^*.

Let us now consider the case $p \in (0,1)$. Denoting the crash hedging strategy in this situation by $\widehat{\pi}_p$ and the corresponding utility function by $\widehat{\nu}_p(t,x) := \nu_{\widehat{\pi}_p}(t,x)$, the defining equilibrium equation for the crash hedging strategy can be written as

$$
\begin{aligned}
& \widehat{\nu}_p(t,x) &=& \; p \cdot \nu_1\left(t, x\left(1-\widehat{\pi}_p(t)k^*\right)\right) + (1-p)\,\nu_{\widehat{\pi}_p}(t,x) \\
\Longleftrightarrow \quad & \widehat{\nu}_p(t,x) &=& \; p \cdot \nu_1\left(t, x\left(1-\widehat{\pi}_p(t)k^*\right)\right) + (1-p)\,\widehat{\nu}_p(t,x) \\
\Longleftrightarrow \quad & \widehat{\nu}_p(t,x) &=& \; \nu_1\left(t, x\left(1-\widehat{\pi}_p(t)k^*\right)\right),
\end{aligned}
$$

hence $\widehat{\pi}_p \equiv \widehat{\pi}$. This result shows that the crash hedging strategy remains the same even if the probability of a crash is known. Thus, this result justifies the wording **worst case scenario** of the above developed concept. This is due to the fact that the worst case scenario should be independent of the probability of the worst case and which has been shown above. Let us summarize this result in a proposition.

Proposition 3.1
Given that the probability of a crash is positive, the worst case scenario portfolio problem as it has been defined in Definition 2.1 is independent of the probability of the worst case.

If the probability of a crash is zero, the worst case scenario portfolio problem reduces to the classical crash–free portfolio problem.

3.3 The q–quantile crash hedging strategy

Obviously, the concept of the worst case scenario has the disadvantage that additional information (namely the given probability of a crash) is not used. However, if the probability of a crash and the probability of the crash size is known, it is possible to construct the **(lower) q–quantile crash hedging strategy**.

Assume that $p_C(t) \in [0, 1]$ is the probability of a crash at time $t \in [0, T]$ and let $p(k, t) \in [0, 1]$ be the density of the distribution function for a crash of size $k \in [k_*, k^*]$ at time t. Moreover, suppose that a function $q : [0, T] \longrightarrow [0, 1]$ is given. With this define

$$k_q\,(t;\pi) := \left\{ \begin{array}{ll} 0 & \text{if } 1-p_c(t) \geq q(t) \\ \inf\left\{ k_q : 1-p_c(t)+p_c(t)\int\limits_{k_*}^{k_q} p(k,t)\,dk \geq q(t) \right\} & \begin{array}{l}\text{if } 1-p_c(t) < q(t) \\ \text{and } \pi \geq 0\end{array} \\ \sup\left\{ k_q : 1-p_c(t)+p_c(t)\int\limits_{k_q}^{k^*} p(k,t)\,dk \geq q(t) \right\} & \text{else} \end{array} \right\}$$

for any given portfolio strategy π. This has the following interpretation. The probability that at most a crash of size $k_q(t)$ at time t happens is $q(t)$. Equivalently, the probability that a crash higher than $k_q(t)$ will happen at time t is less than $1-q(t)$. Obviously, this is a Value at Risk approach.

Notice that the worst case of a nonnegative portfolio strategy is either a crash of size k^* or no crash. On the other hand, the worst case of a negative portfolio strategy is either a crash of size k_* or no crash. Correspondingly, the q–quantile calculates differently for negative portfolio strategies (see the third row) than for the nonnegative portfolio strategies (see the second row). Furthermore, denote by

$$K_q(t) := \left\{ \begin{array}{ll} \{0\} & \text{if } k_q(t) = 0 \\ \{0\} \cup [k_*, k_q(t)] & \text{if } k_q(t) \neq 0 \text{ and } \pi \geq 0 \\ \{0\} \cup [k_q(t), k^*] & \text{else} \end{array} \right\} .$$

Definition 3.2

1. *The problem to solve*

$$\sup_{\pi(\cdot)\in A(x)} \inf_{\substack{0\le\tau\le T,\\ k\in K_q(t)}} \mathbb{E}\left[\ln\left(X^{\pi}(T)\right)\right] , \tag{14}$$

where the final wealth $X^{\pi}(T)$ *in the case of a crash of size* k *at time* s *is given by*

$$X^{\pi}(T) = [1 - \pi(\tau)k]\, X_1^{\pi,\tau,X_0^{\pi}(\tau)}(T) , \tag{15}$$

with $X_1^{\pi,\tau,X_0^{\pi}(\tau)}(t)$ *as above, is called the* **(lower)** q**–quantile scenario portfolio problem**.

2. *The* **value function** to the above problem *is defined via*

$$\nu_q(t,x) = \sup_{\pi(\cdot)\in A(t,x)} \inf_{\substack{t\le\tau\le T,\\ k\in K_q(t)}} \mathbb{E}\left[\ln\left(X^{\pi,t,x}(T)\right)\right]. \qquad (16)$$

3. *A portfolio strategy* $\widehat{\pi}_q$ *determined via the equation*

$$\nu_{\widehat{\pi}_q}(t,x) = \nu_1\left(t, x\left(1-\widehat{\pi}_q(t)k_q(t)\right)\right) \quad \textit{for all } t\in[0,T] \textit{ with } k_q(t)>0$$

will be called a **(lower) q–quantile crash hedging strategy**.

Remark 3.3

1. It is straightforward to see that the 1–quantile scenario portfolio problem is equivalent to the worst case scenario portfolio problem in Definition 2.1. Moreover, the 1–quantile crash hedging strategy is equivalent to the crash hedging strategy in Definition 3.1 in M. (2006), p. 602.

2. Notice that the q–quantile scenario portfolio problem is only a q–quantile concerning the crash. The randomness of the market movement represented in the model by a geometric Brownian motion has been averaged out, namely by taking the expectation – and not the q–quantile.

Define the **support of $\boldsymbol{k_q}$** to be

$$\mathrm{supp}\,(k_q) := \{t \in [0, T] : k_q(t) > 0\}\,.$$

Theorem 3.4

Let us suppose that k_q *is continuously differentiable on supp*(k_q) *with respect to* t.

1. *Then there exists a unique (lower)* q*-quantile crash hedging strategy* $\widehat{\pi}_q$, *which is on supp*(k_q) *given by the solution of the differential equation*

$$\begin{aligned}\widehat{\pi}_q'(t) &= \left(\widehat{\pi}_q(t) - \frac{1}{k_q(t)}\right)\left[\frac{\sigma_0^2}{2}\left(\widehat{\pi}_q(t) - \pi_0^*\right)^2 + \Psi_1 - \Psi_0\right] - \widehat{\pi}_q(t) k_q'(t), \\ \widehat{\pi}_q(T) &= 0\,.\end{aligned}$$

For $t \in [0, T] \setminus$ *supp*(k_q) *set* $\widehat{\pi}_q(t) := \pi_0^*$.

Moreover, the q*–quantile crash hedging strategy is for* $t \in$ *supp*(k_q) *bounded by*

$$0 \le \widehat{\pi}_q(t) < \frac{1}{k_q(t)} \le \frac{1}{k_*} \quad \textit{if } \Psi_1 \ge r_0.$$

Additionally, if $\Psi_1 \leq \Psi_0$ *and* $\pi_0^* \geq 0$*, the* q*–quantile crash hedging strategy has another upper bound with* $\widehat{\pi}_q < \pi_0^* - \sqrt{\frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1)}$.

On the other side, if $\Psi_1 < r_0$ *the* q*–quantile crash hedging strategy is bounded by*

$$\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1)} < \widehat{\pi}_q(t) < 0 \qquad \textit{for } t \in [0, T).$$

2. *If* $\Psi_1 < \Psi_0$ *and* $\pi_0^* < 0$*, there exists a partial* q*–quantile crash hedging strategy* $\tilde{\pi}_q$ *at time* t *(which is different from* $\widehat{\pi}_q$*), if*

$$S_q(t) := T - \frac{\ln\left(1 - \pi_0^* k_q(t)\right)}{\Psi_0 - \Psi_1} > 0 \qquad \textit{for } t \in \textit{supp}\,(k_q)\,. \qquad (17)$$

With this, $\tilde{\pi}_q(t)$ *is given by the unique solution of the differential equation*

$$\begin{aligned}\tilde{\pi}_q'(t) &= \left(\tilde{\pi}_q(t) - \frac{1}{k_q(t)}\right)\left[\frac{\sigma_0^2}{2}\left(\tilde{\pi}_q(t) - \pi_0^*\right)^2 + \Psi_1 - \Psi_0\right] - \tilde{\pi}_q(t)k_q'(t), \\ \tilde{\pi}_q\left(S_q(t)\right) &= \pi_0^*.\end{aligned}$$

For $S_q(t) \leq 0$ set $\tilde{\pi}_q(t) := \pi_0^*$. This partial crash hedging strategy is bounded by

$$\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)} < \tilde{\pi}_q \leq \pi_0^* < 0.$$

If k_q is independent of the time t, the optimal portfolio strategy for an investor, who wants to maximize her q–quantile scenario portfolio problem, is given by

$$\bar{\pi}_q(t) := \min\left\{\hat{\pi}_q(t), \tilde{\pi}_q(t), \pi_0^*\right\} \qquad \text{for all } t \in [0, T], \qquad (18)$$

where $\tilde{\pi}_q$ will be taken into account, if it exists. $\bar{\pi}_q$ will also be called the **optimal q–quantile crash hedging strategy**.

Remark 3.5

1. It is also possible to solve the above problem if k_q is not continuously differentiable. In order to verify this define $\widehat{\pi}_k$ to be the unique solution of

$$\widehat{\pi}_k'(t) = \left(\widehat{\pi}_k(t) - \frac{1}{k}\right)\left[\frac{\sigma_0^2}{2}\left(\widehat{\pi}_k(t) - \pi_0^*\right)^2 + \Psi_1 - \Psi_0\right] \quad \text{and} \quad (19)$$

$$\widehat{\pi}_k(T) = 0, \quad (20)$$

for $k > 0$. Set then $\widehat{\pi}_q(t) := \widehat{\pi}_{k_q(t)}(t)$ where the convention $\widehat{\pi}_0(t) := \pi_0^*$ is used in order to include the case $k_q(t) = 0$. Note that this procedure is also possible for continuously differentiable k_q. However, only if k_q is continuously differentiable, it is possible that $\widehat{\pi}_q$ is also continuously differentiable.

2. Notice that $\widehat{\pi}_{k_1}' < \widehat{\pi}_{k_2}'$ for $k_1 < k_2$. Hence, $\widehat{\pi}_{k_1} \geq \widehat{\pi}_{k_2}$ with strict inequality applying on $[0, T)$. Thus, in particular, $\widehat{\pi}_q(t) > \widehat{\pi}(t)$ for $t \in [0, T)$ for any q which satisfies $q(t) < 1$ for $t \in [0, T)$. Moreover, $\widehat{\pi}_{q_1}(t) \leq \widehat{\pi}_{q_2}(t)$, if $q_1 > q_2$.

3. For this remark, let us suppose that the market conditions do not change, hence $\Psi_1 = \Psi_0$. Moreover, keep in mind that any $\widehat{\pi}_k$ is bounded by π_0^* from above. Thus, it is clear that

$$\psi(t) := \left\{ \begin{array}{ll} 0 & \text{for } t = T \\ \pi_0^* & \text{else} \end{array} \right\}$$

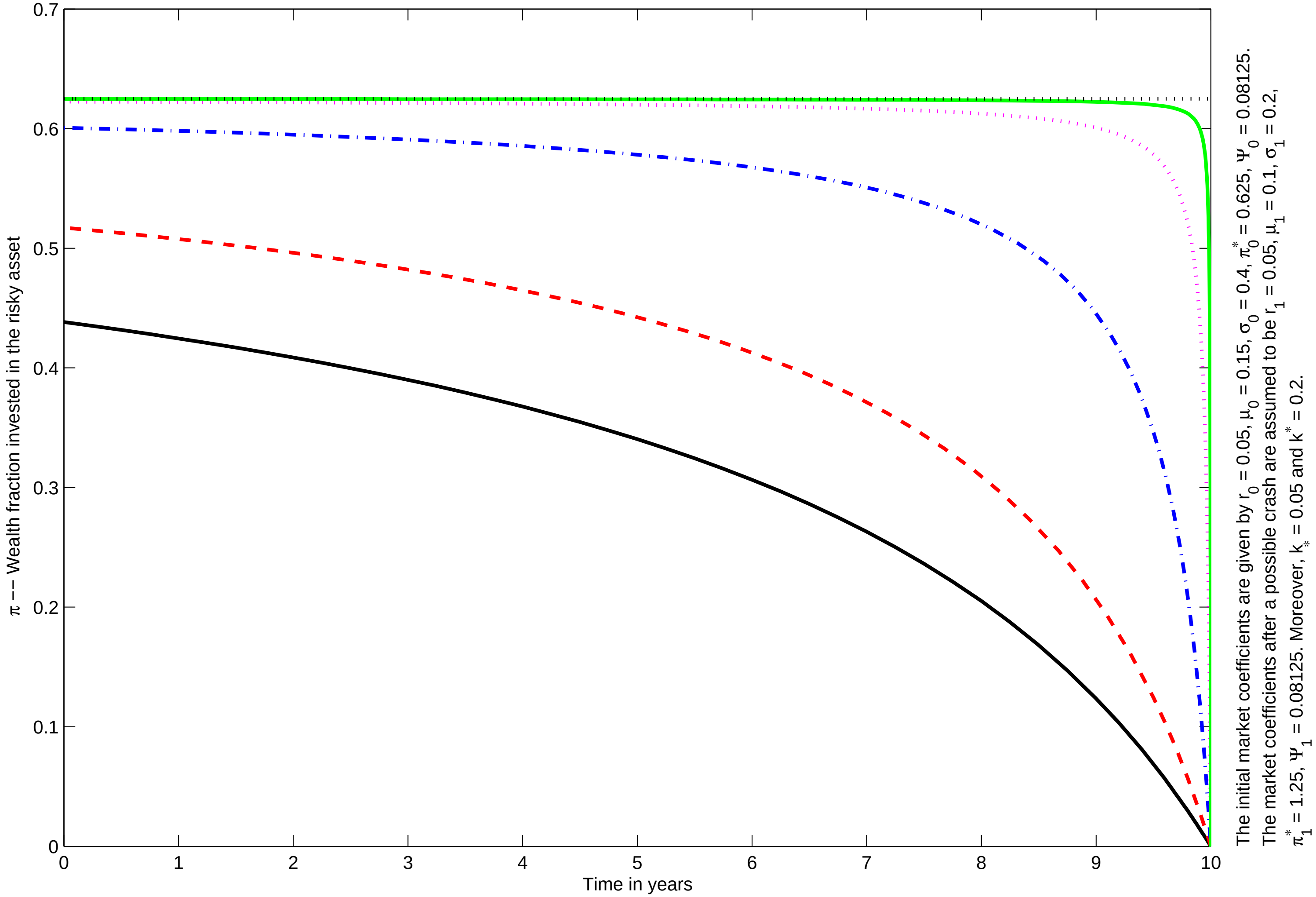
is an upper bound for any $\widehat{\pi}_k$ with $k > 0$. Unfortunately, it is not possible to show that

$$\widehat{\pi}_{k^*} \longrightarrow \psi$$

for $k^* \downarrow 0$ with $k^* \neq 0$, since $\widehat{\pi}_{k^*}$ is only known implicitly and not explicitly. However, this is exactly what can be observed in practice.

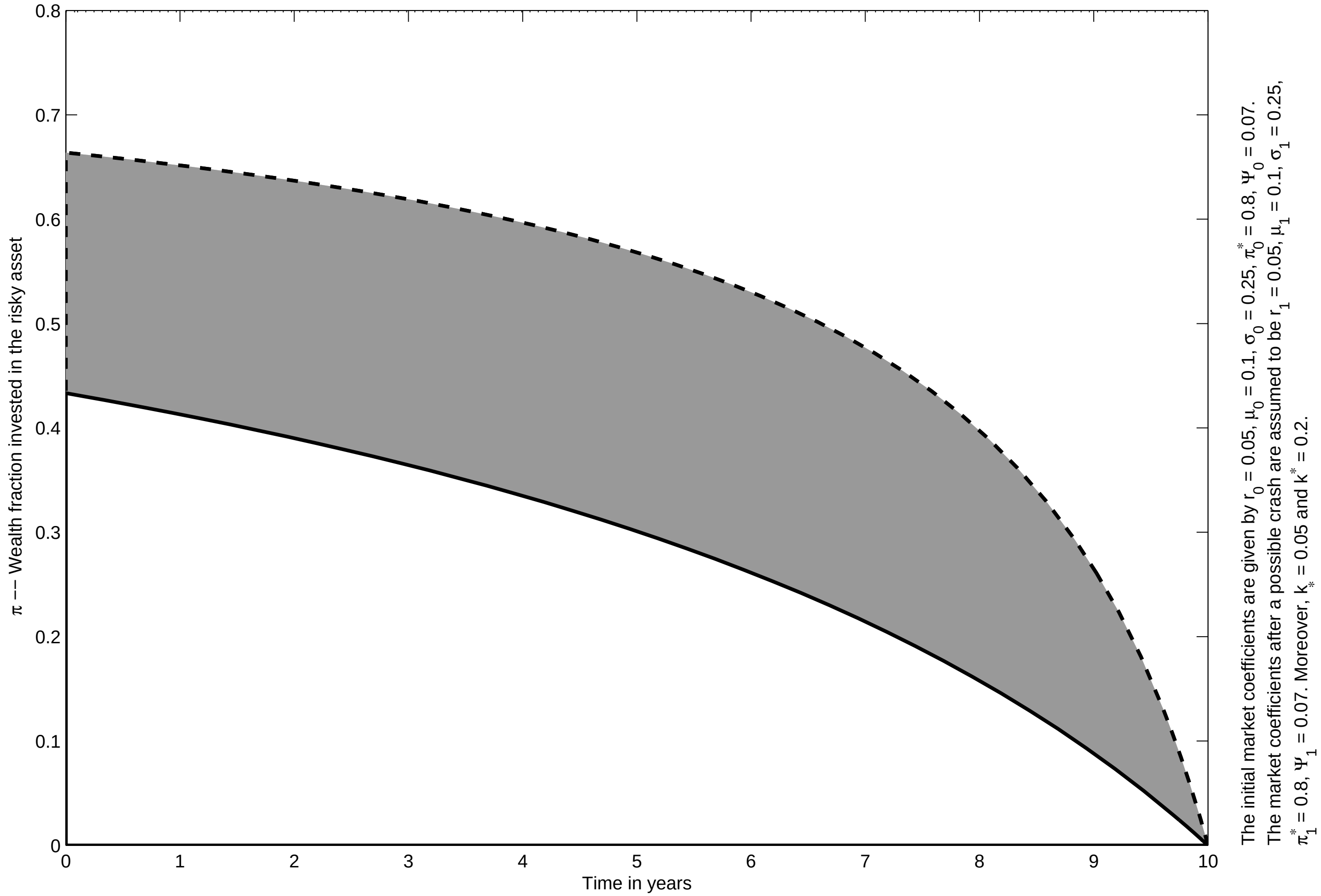
Moreover, keep in mind that the case $k = 0$ yields π_0^* as the optimal portfolio with $\pi_0^* \neq \psi$.

Example of $k \longrightarrow 0$ for $\Psi_1 = \Psi_0$ and $\pi_0^* \geq 0$



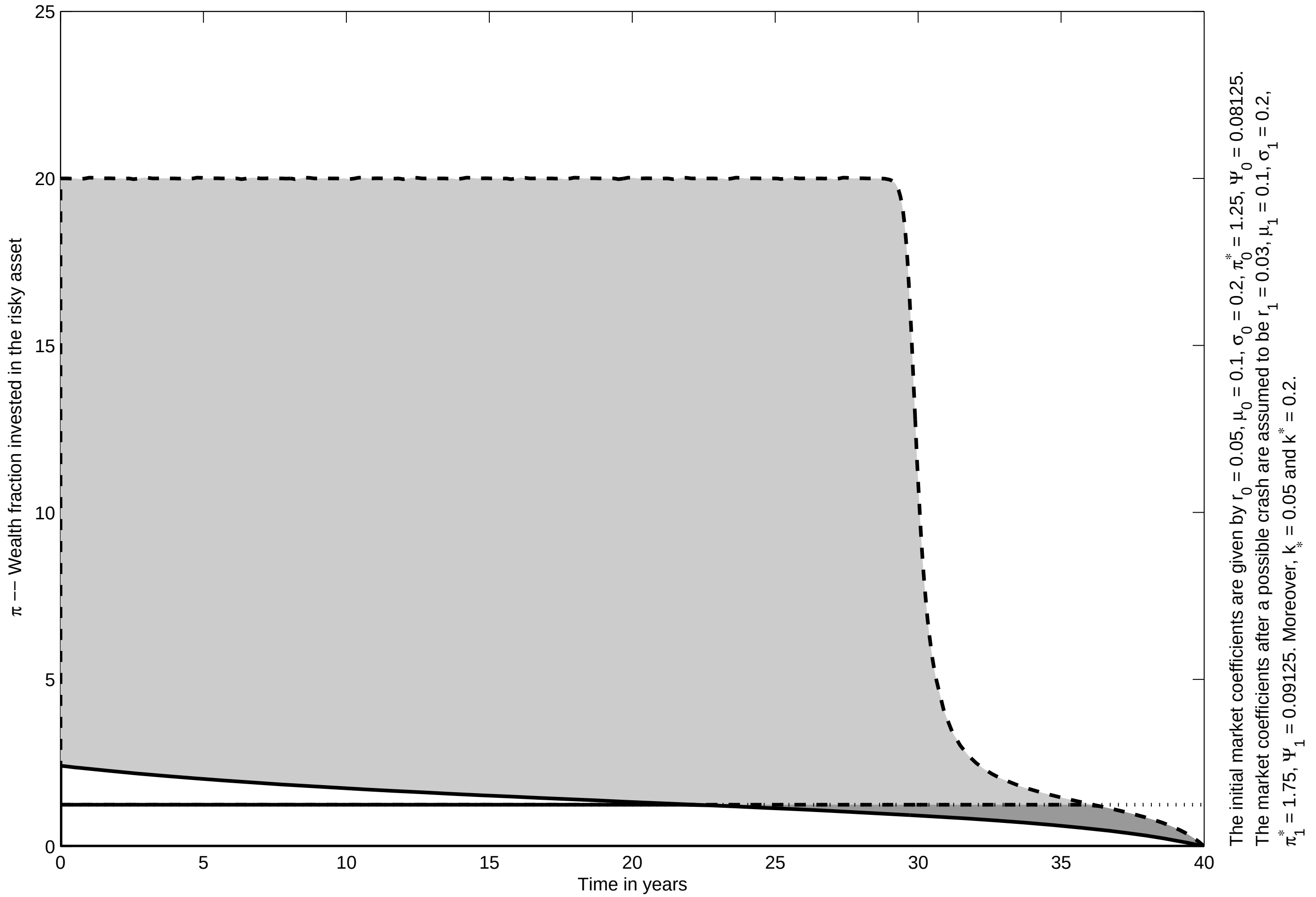
This graphic shows $\widehat{\pi} = \widehat{\pi}_{k^*}$ (black dashed line), $\widehat{\pi}_{\frac{k^*}{2}}$ (red dashed line), $\widehat{\pi}_{\frac{k^*}{10}}$ (blue dash–dotted line), $\widehat{\pi}_{\frac{k^*}{100}}$ (cyan dotted line), $\widehat{\pi}_{\frac{k^*}{1000}}$ (green solid line), and π_0^* (black dotted line).

The Range of (Optimal) q–Quantile Crash Hedging Strategies for $\Psi_1 = \Psi_0$ and $\pi_0^* \geq 0$



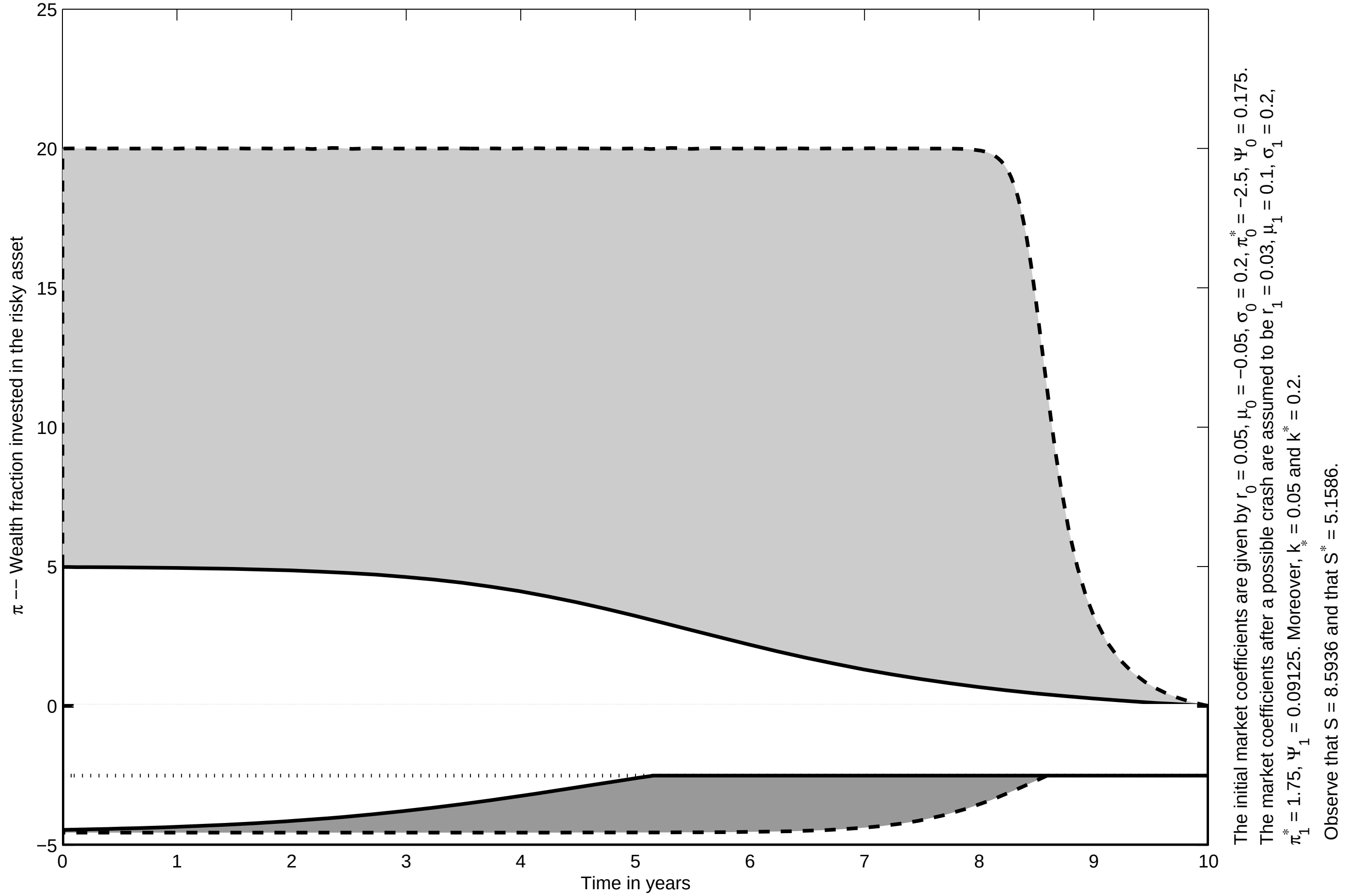
This graphic shows $\widehat{\pi}_{k^*}$ (black solid line), $\widehat{\pi}_{k_*}$ (black dashed line), the range of possible q–quantile crash hedging strategies (light grey and dark grey area), the range of possible optimal q–quantile crash hedging strategies (dark grey area), and π_0^* (black dotted line).

The Range of (Optimal) q–Quantile Crash Hedging Strategies for $\Psi_1 > \Psi_0$ and $\pi_0^* \geq 0$



This graphic shows $\widehat{\pi}_{k^*}$ (black solid line), $\widehat{\pi}_{k_*}$ (black dashed line), the range of possible q–quantile crash hedging strategies (light grey and dark grey area), the range of possible optimal q–quantile crash hedging strategies (dark grey area), and π_0^* (black dotted line).

The Range of (Optimal) q–Quantile Crash Hedging Strategies for $r_0 < \Psi_1 < \Psi_0$ and $\pi_0^* < 0$



This graphic shows $\widehat{\pi}_{k^*}$ and $\bar{\pi}_{k^*}$ (black solid line), $\widehat{\pi}_{k_*}$ and $\bar{\pi}_{k_*}$ (black dashed line), the range of possible q–quantile crash hedging strategies (light grey area), the range of possible optimal q–quantile crash hedging strategies (dark grey area), and π_0^* (black dotted line).

4 Extensions

Possible extensions are

- More crashes (see Korn and Wilmott (2002), Korn and M. (2005), M. (2004)).
  $\Longrightarrow$ System of differential equations.
- More stocks (see e. g. Hua and Wilmott (1997)).
  $\Longrightarrow$ Numerical methods and crash coefficients.
- General utility functions (see Korn and M. (2005), M. (2004)).
  $\Longrightarrow$ Stochastic control approach.

- Connection to problems in actuarial mathematics (see Korn (2005)).
  $\Longrightarrow$ Investing in the presence of additional risk processes.
- Worst case scenario optimization for reinsurance (see Korn, M. , Steffensen, work in progress)
- Costs and benefits of crash hedging (see M. (2004)).
  $\Longrightarrow$ Calculating the costs and the potential benefits of crash hedging.
- Differential games (see Korn and Steffensen (2005))
- Market coefficients after a crash depend on the crash size k (see M. (2004)).
  $\Longrightarrow$ Differential equations for $\widehat{\pi}$ and $\widehat{k}$, the worst case crash size.

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