Using Economic Contexts to Advance in Mathematics

Erik Balder

University of Utrecht, Netherlands

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Background

Evidence shows basic mathematical skills to be important for students' performance at intermediate micro and macro level

This explains why undergraduates in economics take one or more courses of the type "maths for economics"; minimally these cover some calculus (preferably also multivariate), linear algebra and probability

However, research-oriented undergraduates could handle more than just these basic subjects, which make them miss out in particular on analytical skills (including the capacity to understand and produce proofs) that are essential for several graduate programmes

This talk discusses some problems with the present structure for those students and suggests alternatives

Current problems with standard ”maths for economics”

1. If taught by mathematicians: as a rule, both course material and teacher disregard most aspects of economics because of ignorance of or disinterest in economic contexts. Essentially, what they teach is “maths for *physical sciences*”

2. If taught by economists: all too direct (or even erroneous) adoption of material from mathematical textbooks. This invariably ignores some unique aspects of economic models (e.g. nonnegativity of variables like quantities or prices, concavity, monotonicity)

Consequently, students can only partially explore economic contexts and then they usually do so uncritically, because some incisive mathematical insights and tools have not been taught

E.g., in optimization/micro delicate boundary aspects are avoided by a priori exclusion of models with corner solutions (this ”explains” absence of linear utility functions ...), non-differentiability of Leontiev functions is side-stepped by jumping to a graphical plot, shut-down price is often not presented as a corner solution, etc.

So both educational arrangements 1. and 2. tend to produce a cookbook-cum-recipe approach to ”maths for economics”, which leaves little room for the understanding of underlying reasoning or proofs

While this may still be acceptable for generic majors, it represents a definite handicap for research-oriented undergraduates in economics, the target group of this talk

To compound these problems: even if supplementary math courses (e.g., multivariate calculus, (geometric) linear algebra, optimization, convex analysis, ...) are available, they often demand too much time from a senior undergraduate and lack of training in mathematical reasoning may be another obstacle

In both arrangements it is not sufficiently realized by economics departments that mathematics, even in basic “maths for economists” courses, could be tweaked for greater educational benefit, in particular to those students

Some possible improvements

Some simple possible improvements of basic "maths for economics" courses (non-traditional subjects from a math perspective):

- Already scalar optimization should spend time on optimization over *intervals* (preferably parametrized), so as to handle boundary optima ("corner solutions") early on
- *Parametrized* problems (e.g. optimization or differential equations, etc.) should be practised at an early stage; e.g. think of influence of exogenous variables on endogenous ones (i.e. outcomes). This leads to student being able to distinguish different cases giving different outcomes. With parameter dependence shown by outcomes one can practice (monotonicity, limits ...) and then match this with economic intuition
- Systems of *inequalities* can be just as important for economics as more standard systems of equalities (e.g. linear equalities); e.g. think of linear programming or mathematical finance models

“maths-plus within economics”: a catch-up course for research-oriented undergraduates

Outline of a course “maths-plus” that better prepares undergraduate students of economics for graduate school

Mathematical subjects treated (within coherent economic contexts): linear algebra, multivariate calculus and optimization, convex analysis

Note: hardly any attention for stochastics, because of time constraints

The course trains students both in calculations and in reasoning/proofs

Coherent accounts within economics, interspersed with mathematical intermezzi – contrast with “maths for economics”

On many occasions student is invited to complete proofs, produce counterexamples, etc.

Important role for self-testing for correctness of outcomes of computations – this facilitates self-study

Examples of such self-testing

Approximate correctness of calculated least squares estimate can be seen from data-plot; complete correctness can be seen by verifying normal equations

Correctness of no-arbitrage price of a contingent claim can be checked by duality arguments with risk-neutral probabilities

Correctness of calculated Marshallian demand function can be verified by (*i*) validity of Roy's identity, (*ii*) limit behaviour when prices or income go to infinity, (*iii*) validity of homogeneity, (*iv*) graphical solution methods, etc.

Correctness of calculated Marshallian and Hicksian demand functions can be checked by verifying duality relationships

Outline of “maths-plus within economics”

1. Least squares method requires intermezzi with linear algebra (column space, linear independence, orthogonality, etc.)

2. Connect least squares method with orthogonal projections and best approximation (of a column space)

3. Optional: via least squares modification move to best approximation of convex cones; yields Farkas’ lemma, separation of convex sets, etc.

To illustrate background of such connections, consider picture of best approximation of vector $\mathbf{x}$ by convex set S, with associated obtuse angle property:

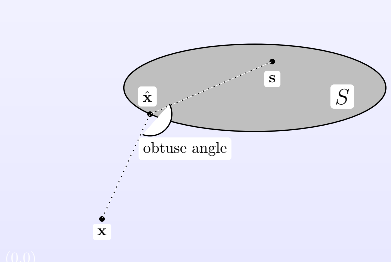

Figure: $(\mathbf{x} - \hat{\mathbf{x}}) \cdot (\mathbf{s} - \hat{\mathbf{x}}) \leqslant 0$ FOR ALL $\mathbf{s} \in S$

4. Optional: apply 3. to no-arbitrage pricing in simple mathematical finance model (finitely many states of nature)

5. Study of consumer demand: full study requires careful study of optimization, including some convex analysis and a *bespoke* Kuhn-Tucker theorem for utility maximization and expenditure minimization. It deals with hitherto overlooked (!) peculiar properties of utility functions: e.g., even the simple $\sqrt{x_1 x_2}$ (Cobb-Douglas) or $(\sqrt{x_1} + \sqrt{x_2})^2$ (CES) utility functions are non-differentiable in points where $x_1 = 0$ or $x_2 = 0$

A powerful associated optimization method[1] serves as generator of concrete Marshallian and Hicksian demand functions. This yields rather non-trivial calculations, whose outcomes can be subjected to the aforementioned self-testing, such as:

[1] See reference at end of presentation

- checking the validity of Roy's identity practices multivariate differentiation skills
- matching limit behaviour of demand with economic intuition when, say, prices or income go to infinity practices limit taking skills
- checking the homogeneity of demand function practices algebraic skills
- checking the validity of duality relationships or Slutsky's decomposition practices skills to substitute, differentiate and reason logically

Comparable comments apply to production theory

6. For completeness, some attention could be given to other subjects (e.g. partial equilibrium computation), but their marginal contribution to practising the above-mentioned skills is small

Three examples of exercises

To test the understanding of least squares:

Exercise 1. Consider the constrained least squares problem

$$\text{minimize} \|\mathbf{y} - X\boldsymbol{\theta}\|^2 \text{ over all } \boldsymbol{\theta} \in \mathbb{R}^d \text{ such that } A\boldsymbol{\theta} = \mathbf{0}.$$

X design matrix (full column rank), matrix A

Prove, by using Pythagoras' theorem that the optimal solution is given by

$$\tilde{\boldsymbol{\theta}} = (X^T X)^{-1} X^T \mathbf{y} - (X^T X)^{-1} A^T [A(X^T X)^{-1} A^T]^{-1} A(X^T X)^{-1} X^T \mathbf{y}$$

after showing that $A(X^T X)^{-1} A^T$ is invertible and that $A\tilde{\boldsymbol{\theta}} = \mathbf{0}$

Hint: imitate the Pythagoras-based proof of LSE in the main text, but do not substitute the longish expression on the right too soon.

Comments about solving Exercise 1:

Proving invertibility imitates and extends the proof of invertibility of $X^T X$ that the student has digested

As for the use of Pythagoras, next to finding the correct imitation, which requires insight in the nature of the least squares estimator, the student must prove the orthogonality needed to apply Pythagoras, i.e.

$$(\mathbf{y} - X\tilde{\boldsymbol{\theta}}) \cdot X(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}) = 0$$

must be proven for every $\boldsymbol{\theta}$ with $A\boldsymbol{\theta} = \mathbf{0}$. This certainly requires some algebraic skills because of formula for $\tilde{\boldsymbol{\theta}}$ is so involved. Of course this also produces the normal equations for this linear model

A more challenging version of this exercise would ask the student to come up with the proper formula for $\tilde{\boldsymbol{\theta}}$ (say by Lagrange's FONC) and then continue with the questions asked in the exercise

To test understanding of no-arbitrage pricing:

Exercise 2. Consider a financial one-period model with state space $\Omega := \{\omega_1, \omega_2, \omega_3\}$ and three assets. The first asset (index = 0) is risk-less and carries interest $r = 0$. The second asset (index = 1) has value $W_1(t) = 4$ at time $t = 0$ and values $W_1(t)(\omega_1) = 8$, $W_1(t)(\omega_2) = 6$ and $W_1(t)(\omega_3) = 3$ at time $t = 1$. For the third asset these values are $W_2(0) = 7$ and $(W_2(\omega_1), W_2(\omega_2), W_2(\omega_3)) = (10, 8, a)$, where $a \in \mathbb{R}$ is a parameter.
(i) Determine for every $a \in \mathbb{R}$ the (possibly empty) set of all risk neutral probability vectors
(ii) Determine the set A of all $a \in \mathbb{R}$ for which an arbitrage opportunity exists
(iii) Determine for every $a \in A$ precisely one concrete (but still in terms of a of course) arbitrage opportunity that has value zero at $t = 0$
(iv) Determine for every $a \in \mathbb{R}$ the set of all replicable (= hedgeable) contingent claims (C_1, C_2, C_3) at $t = 1$.

Comments about solving Exercise 2:

Here the student must understand the basics of the dual relationship between no-arbitrage and risk-free probability vectors

Part (*iii*) requires the student to come up with a concrete solution of a system consisting of one linear equality and three linear inqualities

Exercise 3. Preferences of a consumer can be represented by $u(x_1, x_2) = \min(\sqrt{x_1}, x_2 - \beta)$ on $\mathbb{R}^2_+$. Here $\beta \geqslant 0$ is a parameter.

(i) Determine for $\beta = 2$ the indifference set that contains the bundle $(4, 6)$ and also the indifference set containing $(9, 1)$.

(ii) Use the graphical solution method to determine the *formula* (so do not just indicate points in a picture) of the Marshallian demand $(x_1^m(p_1, p_2, y), x_2^m(p_1, p_2, y))$ of the consumer for all values $p_1, p_2, y > 0$.

(iii) Determine $(x_1^m(p_1, p_2, y), x_2^m(p_1, p_2, y))$ also entirely analytically, by following the utility maximization solution method. Check that your outcome is the same as in (ii).
(iv) Test your outcome in (ii)-(iii) by computing its limit for $\beta \to \infty$ and by verifying if this limit behaviour matches your economic intuition.

Comments about solving Exercise 3:

(i) helps the student on his/her way in part (ii)

The desired bundle is graphically seen to be (1) the intersection of $x_2 = \sqrt{x_1} + \beta$ and the budget line $p_1 x_1 + p_2 x_2 = y$ if $\beta \leqslant y/p_2$ and (2) the corner point $(0, y/p_2)$ if $\beta > y/p_2$

In case (1) the student must see that he/she can employ the standard solution formula for solving quadratic equations after substituting $z_1 := \sqrt{x_1}$. The ensuing algebra is not entirely trivial, as can be seen from the formulas for the outcome:

$$x_1^m = (-\frac{p_2}{2p_1} + \frac{1}{2p_1}\sqrt{p_2^2 + 4p_1(y - p_2\beta)})^2,$$

$$x_2^m = -\frac{p_2}{2p_1} + \beta + \frac{1}{2p_1}\sqrt{p_2^2 + 4p_1(y - p_2\beta)},$$

As mentioned before, such heavily parametrized formulas open the way to several self-tests and questions. Here part (iii) is kept simple: for $\beta \to \infty$ only case (2) prevails, which has $(0, y/p_2)$ as the optimal bundle.

For more information and critical evaluations of current literature:

www.staff.science.uu.nl/~balde101/#edu

For more information regarding footnote 1:

www.staff.science.uu.nl/~balde101/giffenvol.pdf