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Optimal Portfolios, Part II: New Variations of an Old Theme

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1. Modifications and generalizations
2. Optimal investment with derivatives
3. Optimal investment with defaultable securities
4. Optimal investment with crashes and unhedgeable risks
5. Optimal investment with transaction costs
6. Optimal investment with stochastic interest rates

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1. Modifications, recent generalizations, open problems

Theory we have learned so far is nice, but there are many aspects not dealt with

- ***Transaction costs*** (fixed and proportional transaction costs)
  $\Rightarrow$ Problem: hard to solve, ***curse of dimensionality***
- Investment with bonds, .. (i.e. ***stochastic interest rates***):
  $\Rightarrow$ Problem: ***Additional interest rate risk***, counter examples
- Investment with ***derivatives***
  $\Rightarrow$ Problem: ***Non-linear portfolios***
- Investment with ***defaultable securities***
  $\Rightarrow$ Problem: ***Additional default risk***
- Investment with ***crashes***
  $\Rightarrow$ Problem: No full probabilistic information, ***worst-case control***
- **.......**

2. Optimal investment with derivatives

Aims and Results

Motivation :

Change of roles in option trading

- Option portfolios for hedging purposes $\rightarrow$ Control of the "Greeks"
- Option portfolios for speculative reasons $\rightarrow$ **?**

Suggestion :

Maximise expected utility of terminal wealth of option portfolio :

$$\max_{\varphi} E\left(U\left(X\left(T\right)\right)\right)$$

$$X(t) = \varphi_0(t) P_0(t) + \sum_{i=1}^{n} \varphi_i(t) f^{(i)}\left(t, P_1(t), .., P_n(t)\right)$$

"Wealth process"

Goal : Determination of optimal option portfolios

Theorem (K., Trautmann (1998))

Given the "Delta matrix" $\psi(t) = (\psi_{\mathrm{ij}}(t))$, i, j = 1, ..., n, with

$$\psi_{ij}(t) := f^{(i)}{}_{p_j}(t, P_1(t), \ldots, P_n(t)), \quad t \in [0, T)$$

is regular, the option portfolio problem (O) has the following explicit solution:

a) The optimal terminal wealth B^* ***coincides*** with the optimal terminal wealth of the corresponding stock portfolio problem.

b) Let $\xi(t)$ be the optimal trading strategy of the correponding stock portfolio problem. Then, the optimal trading strategy $\varphi(t)$ for the option portfolio problem is given by

$$\overline{\varphi}(t) = (\psi^{\mathrm{T}}(t))^{-1} \overline{\xi}(\mathrm{t}),$$

$$\varphi_0(t) = \left(X(t) - \sum_{i=1}^{n} \varphi_i(t) P_i(t)\right) / P_0(t),$$

where $\overline{\varphi}(t)$, $\overline{\xi}(t)$ are the last n components of $\varphi(t)$ and $\xi(\mathrm{t})$.

Example 1. "Logarithmic utility"

$U(x) = \ln(x)$, $n = 1$

$\Rightarrow$

$$\varphi_1(t) = \frac{b-r}{\sigma^2}\frac{X(t)}{\psi_1(t)P_1(t)}$$

$$\pi_{\text{opt}}(t) := \frac{\varphi_1(t)f(t,P_1(t))}{X(t)} =$$

$$\frac{b-r}{\sigma^2}\frac{f(t,P_1(t))}{f_p(t,P_1(t))P_1(t)}$$

Computational effort

- Comparable to stock portfolio problem

Specific example:

European call option on the stock with

$r = 0$, $b = 0.05$, $\sigma = 0.25$, $T = 1$, $t = 0$, $K = 100$

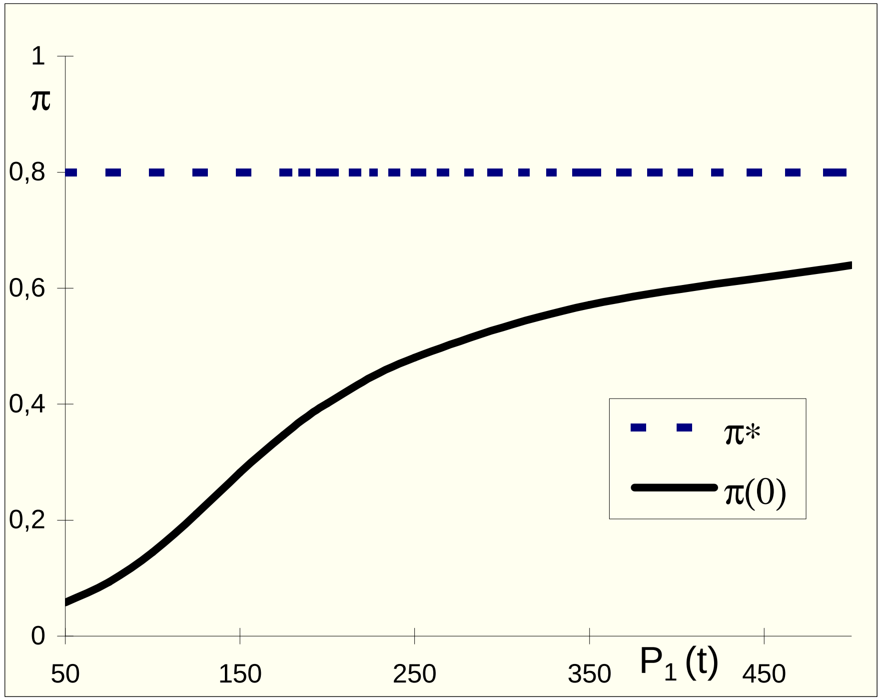


Fig. 1: π^* and $\pi_{\text{opt}}(0)$ as functions of initial stock price

3. Optimal investment with defaultable securities

Merton model: **firm's value** follows a geometric Brownian motion

$$dV = aV\,\mathrm{dt} + \sigma V dW(t)$$

- company Z has issued 1 share and a zero coupon bond with notional B

Result:

- at time T the value of the S shares is given as $(V(T) - B)^{+}$
- at time T the value $B(T)$ of all zero bonds is given as $min(V(T), B)$

i.e. the value of the shares and that of the corporate bond can be got as the prices of call options or as $(B -)$ put options on the firm's value with strike B, respectively.

Black-Scholes-Formula $\Rightarrow$

$$B^{(S)}(t) = Be^{-r(T-t)}\Phi(d_2(t)) + V(t)\Phi(-d_1(t)) \qquad \textit{Corporate bond price}$$

with $$d_1(t) = \frac{ln\left(V(t)/B\right) + \left(r + 1/2\,\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2(t) = d_1(t) - \sigma\sqrt{T-t}$$

Solution method:

Solve the portfolio problem as if the firm value would be tradable !

Proposition (K., Kraft 2001)

If an investor is allowed to invest into the money market account and in the bond issued by the company then his optimal portfolio process is given by

$$\pi^*(t) = \begin{cases} \dfrac{a-r}{\sigma^2} \dfrac{B(t)}{\Phi(-d_1(t))V(t)}, & for \;\; U(x) = ln(x) \\ \dfrac{a-r}{\sigma^2(1-\gamma)} \dfrac{B(t)}{\Phi(-d_1(t))V(t)}, & for \;\; U(x) = \frac{1}{\gamma} x^\gamma \end{cases}.$$

Remark:

- Optimal final wealth is the same as if we could invest in the firm value
- Invested money in the company's bond exceeds that which would be invested in the firm value

4. Optimal investment with crashes and unhedgeable risks:

Alternative crash modelling:

1. Hua & Wilmott (1997): "Number and size of crashes in a given time intervall are bounded" $\Rightarrow$ **no** probabilistic assumptions on height, number and times of occurence of crashes. .

2. K. & Wilmott (2001): "Determine worst-case bounds for the performance of optimal investment".

For simplicity: One bond, one stock, at most one crash in [0, *T*] with a maximal height of $k^* < 1$. Security prices (in "**normal times**"):

$$dP_0(t) = P_0(t) r\, dt\,, \qquad P_0(0) = 1, \qquad \text{“bond”}$$

$$dP_1(t) = P_1(t)(b\, dt + \sigma\, dW(t))\,, \qquad P_1(0) = p\,, \qquad \text{“stock”}$$

At crash time: stock price falls by a factor of $k \in [0, k^*]$

Consequence:

The wealth process $X^\pi(t)$ at crash time satisfies:

$$X^\pi(t-) = (1-\pi(t))X^\pi(t-) + \pi(t)X^\pi(t-)$$

$$\Rightarrow (1-\pi(t))X^\pi(t-) + \pi(t)X^\pi(t-)(1-k) = X^\pi(t-)(1-\pi(t)k) = X^\pi(t)$$

Thus:

Following the portfolio process $\pi(.)$ if a crash of size k happens at time t leads to a final wealth of

$$X^{\pi}(T) = (1 - \pi(t)k)\tilde{X}^{\pi}(T)$$

if $\tilde{X}^{\pi}(.)$ denotes the wealth process in the model without any crash.

Hence:

• "high" values of $\pi(.)$ lead to a high final wealth if no crash occurs at all, but to a high loss at the crash time

• "low" values of $\pi(.)$ lead to a low final wealth if no crash occurs at all, but to a small loss (or even no loss at all !!) at the crash time

Moral:

We have two competing aspects ("Return and insurance") for two different scenarios ("Crash or not") and are therefore faced with a **balance problem between risk and return.**

Aim: Find the best uniform **worst-case bound**, i.e. solve

$$\text{(WP)} \qquad \sup_{\pi(.)\in A(x)} \ \inf_{0\le\tau\le T,\,0\le\kappa\le k^*} E\left(U\left(X^{\pi}(T)\right)\right)$$

where the final wealth satisfies $X^{\pi}(T) = (1-\pi(t)k)\tilde{X}^{\pi}(T)$ in the case of a crash of size k at time t.

Assumption: $b > r$

Note: To avoid bankruptcy we require $\pi(t) < 1/k^*$

Important Remarks: (!)

a) We do **not** (!) compare two different strategies **scenario-wise**. We look separately at the worst-case for both strategies which then yields the worst-case bound. Typically two different strategies have two different worst-case scenarios

b) As we have $b > r$, we do not have to consider portfolio processes $\pi(t)$ that can attain negative values as the log-utility function is increasing in x.

Two extreme strategies (in the log-utility case):

i) $\pi(t) \equiv 0$: **“Playing safe”**

$\Rightarrow$ worst-case scenario: no crash (!) , leading to the following worst-case bound of

$$WCB_0 = E\left(ln\left(X^0(T)\right)\right) = ln(x) + rT$$

ii) $\pi(t) \equiv \pi^* := \dfrac{b-r}{\sigma^2}$: **“Optimal investment in the crash-free world ”**

$\Rightarrow$ worst-case scenario: a crash of maximum size k* (at an arbitrary time instant !) , leading to the following worst-case bound of

$$WCB_{\pi^*} = E\left(ln\left(X^{\pi^*}(T)\right)\right) = ln(x) + rT + \tfrac{1}{2}\left(\tfrac{b-r}{\sigma}\right)^2 T + ln(1-\pi^* k^*)$$

Insights:

- it depends on time to maturity which one of the above strategies is better
- a constant portfolio process **cannot** be the optimal one
- strategy i) takes too few risk to be good if no crash occurs while strategy ii) is too risky to perform well if a crash occurs $\Rightarrow$ the optimal strategy should balance this out !

Theorem 1 **"Dynamic programming principle"**

If $U(x)$ and $v_0(t,x)$ ("value function without crash") are strictly increasing in x then we have

$$v_1(t,x) = \sup_{\pi(.)\in A(t,x)} \inf_{t\le s\le T} E\left(v_0\left(s, \tilde{X}^{\pi}(s)\left(1-\pi(s)k^*\right)\right)\right)$$

Theorem 2 **"Dynamic programming equation"**

Let the assumptions of Theorem 1 be satisfied, let $v_0(t,x)$ be strictly concave in x, and let there exist a continuously differentiable (with respect to time) solution $\hat{\pi}(t)$ of

$$(v_0)_t+(v_0)_x\left(r+\hat{\pi}(t)(b-r)\right)x+\tfrac{1}{2}(v_0)_{xx}\sigma^2\hat{\pi}(t)^2x^2-(v_0)_x x\frac{\hat{\pi}'(t)}{\left(1-\hat{\pi}(t)k^*\right)}k^*=0,\quad \hat{\pi}(T)=0$$

Assume further that we have:

(A) $f(x,y;t):=(v_0)_x(t,x)\left(\left(y-\hat{\pi}(t)\right)(b-r)\right)x+\tfrac{1}{2}(v_0)_{xx}(t,x)\sigma^2\left(y^2-\hat{\pi}(t)^2\right)x^2$

is a concave fuction in (x, y) for all $t\in[0, T)$.

(B) $E^{0,x}\left(\hat{v}\left(t,\tilde{X}^{\pi}(t)\right)\right)\le E^{0,x}\left(\hat{v}\left(t,\tilde{X}^{\hat{\pi}}(t)\right)\right)$ and $E^{0,x}\left(\pi(t)\right)\ge\hat{\pi}(t)$

for some $t\in[0, T), \pi\in A(x)$ $\Rightarrow$ $E^{0,x}\left(v_0\left(t,\tilde{X}^{\pi}(t)\left(1-\pi(t)k^*\right)\right)\right)\le E^{0,x}\left(\hat{v}\left(t,\tilde{X}^{\hat{\pi}}(t)\right)\right).$

$\Rightarrow$ $\hat{\pi}(t)$ is the optimal portfolio ***before*** the crash, $\pi^*(t)$ is the optimal one afterwards

Corollary 3 "Optimal investment in the presence of a crash with log-utility"

There exists a strategy $\hat{\pi}(.)$ such that the corresponding expected log-utility after an immediate crash equals the expected log-utility given no crash occurs at all. It is given as the unique solution $\hat{\pi}(.) \in \left[0, \frac{1}{k^*}\right)$ of the differential equation

$$\pi'(t) = \frac{1}{k^*}\left(1-\pi(t)k^*\right)\left(\pi(t)(b-r) - \tfrac{1}{2}\pi(t)^2\sigma^2 + \tfrac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2\right)$$

$$\pi(T) = 0.$$

Further, this strategy yields the **highest worst-case bound** for our problem (WP).
In particular, this bound is active at each future time point ("uniformly optimal balancing").

After the crash has happened the optimal strategy is given by

$$\pi(t) \equiv \pi^* := \frac{b-r}{\sigma^2} \ .$$

Example 1: ***Logarithmic utility***

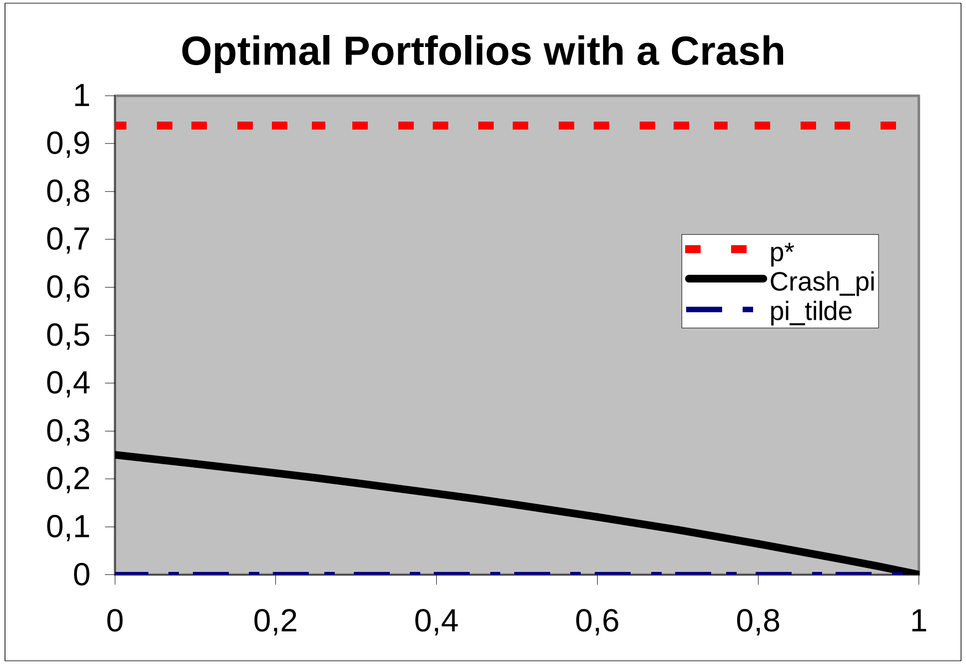
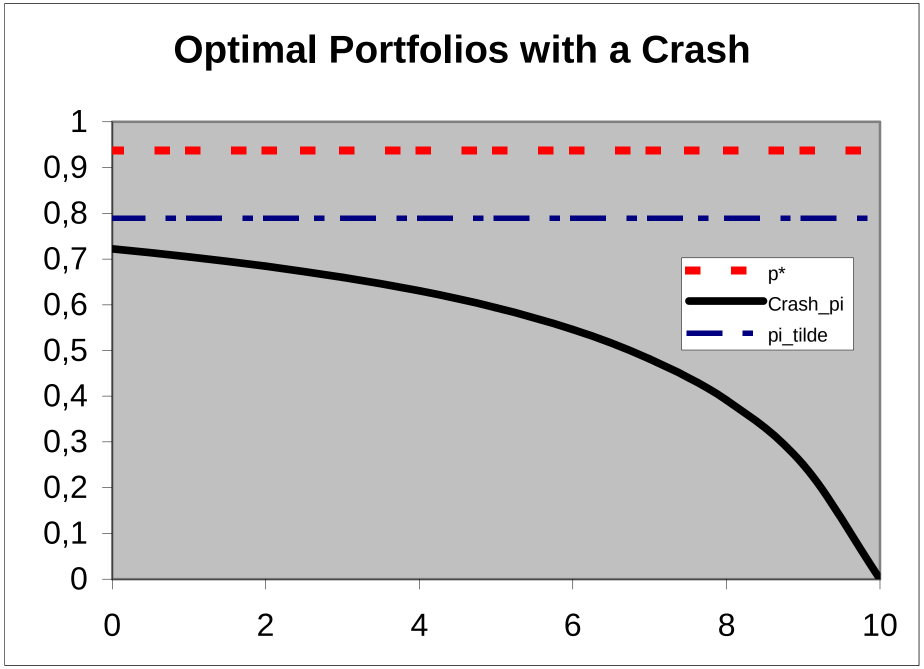


Figure 2: $\hat{\pi}(t)$ as a function of time for the data $b = 0.2$, $r = 0.05$, $\sigma = 0.4$, $k^* = 0.2$, $T = 1$.

Example 2: ***Logarithmic utility***

Same data as before but with a time horizon of T=10:



Interpretation :

• The longer the time to the trading horizon the more attractive it is to invest in the stock, and even a "moderate crash" is no real threat.

• If the final time is near it is good to save the gains (i.e. reduce stock investment) as then there is not enough time to compensate the effect of a crash by an optimal stock investment afterwards.

Generalizations:

- Worst-case investment with insurance risk (K. 2004)
- More than one crash, multiple assets (K., Wilmott (2001)
- Changing market coefficients after a crash (K., Menkens (2004),(2005))
- General utility function and Bellman systems (K., Steffensen (2008))
- Crash hedging and worst-case control (Menkens (2005))

Open problems

- Use of options
- Application to the control of social or technical systems with possible catastrophes

5. Optimal investment with Transaction Costs

Very often: Optimal portfolio processes are **constant** ones. However, this requires trading **at each time instant** !

In the presence of transaction costs $\Rightarrow$ (**immediate) *ruin* !**

Necessary: Consider different class of trading strategies

$\Rightarrow$

1.) Davis/Norman (1990) “Proportional t.c.” : ***Singular stochastic control***

2.) Eastham/Hastings (1988), Korn (1998) “Fixed and prop. t.c.”: ***Impulse control***

3.) Morton/Pliska (1995) “Non-sensical t.c.”: ***Optimal stopping***

Main problem: Practical realization !

The Morton-Pliska Approach (1995)

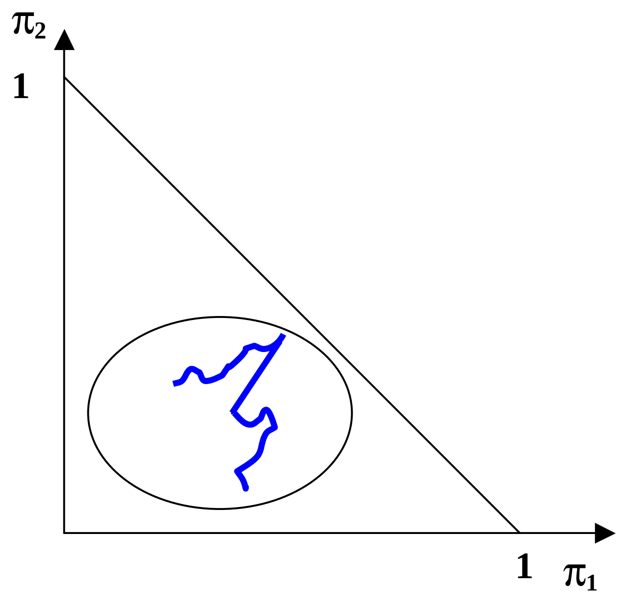
Assumptions:

- Each transaction costs a fixed percentage ε of the investor's wealth
- Find the portfolio process that maximizes the long term growth rate of wealth, i.e. solve

$$\max_{\pi} \lim_{T\to\infty} \frac{1}{T} E\left(ln\left(X^{\pi}(T)\right)\right)$$

Solution:

- Morton and Pliska (1995): A stationary optimal control exists (characterized by a set of variational inequalities) "***Let the portfolio process evolve freely inside a fixed region E, transact back to a fixed point π* inside E***"



The underlying mathematical problem:

With the definition of the operator A via

$$(Ah)(\pi) := \tfrac{1}{2}\sum_{i,j=1}^{n} h_{ij}(\pi)\pi_i\pi_j(e_i' - \pi')\sigma\sigma'(e_j - \pi) + \sum_{i=1}^{n} h_i(\pi)\pi_i(e_i' - \pi')(b - r\underline{1} - \sigma\sigma'\pi)$$

the value function of the Morton-Pliska problem is given as the (smallest) solution of the **variational inequalities**

$$\boxed{\begin{aligned} (Af)(\pi) &\le R - r \\ f(\pi) &\ge -ln(1 - \underline{1}'\pi) \\ \big((Af)(\pi) - R + r\big)\big(f(\pi) + ln(1 - \underline{1}'\pi)\big) &= 0 \end{aligned}}$$

on the unit simplex (with $R = r + \tfrac{1}{2}\pi'\sigma\sigma'\pi$).

A practical advantage of the Morton-Pliska approach:

Possibility to solve multi-asset problems

=>

Atkinson and Wilmott (1995):

Solve the problem via asymptotic expansion

$$\tilde{\pi} := \tilde{\pi}(t) = (\sigma\sigma')^{-1}(b - r\underline{1})$$ “**Merton point**”

$$\tilde{\pi} + \varepsilon^{1/4} C$$ “**continuation region E** ”

$$r^* = r + \tfrac{1}{2}\pi'\sigma\sigma'\pi - 2\varepsilon^{1/2} Tr(HM)$$ “**asymptotic growth rate** ”

with $C := \{\pi \in IR^n \mid \pi' M\pi \leq 1\}$ and M the unique positive solution of the matrix equation

(ME) $$8MHM + 4Tr(HM)M = \sigma\sigma',$$

with $$H_{ij} = \tilde{\pi}_i \tilde{\pi}_j (e_i - \tilde{\pi})'\sigma\sigma'(e_j - \tilde{\pi}), \quad i, j = 1,..,n.$$

However:

The structure of the transaction costs (**always pay a fraction of total wealth independent of the volume of the transaction**) is still non-sensical !

<u>Proposition</u> "*A Second Look at the Morton-Pliska-Approach*" (K. (2004))

Let M, C be as above, $0 < k < 1$ determine the ***real world*** prop. transaction costs. Then we have:

i) Let $y^* \in IR^n$ solve $\max_{y \in IR^n : y'My=1} \sum_{i=1}^{n} |y_i| = \max_{y \in IR^n : y'My=1} \|y\|_1$.

Then, the ***actual transaction costs*** caused by a rebalancing of the holdings when following the Morton-Pliska strategy satisfy

$$X(t-) - X(t) \leq k^* \varepsilon^{1/4} \|y^*\|_1 X(t) =: \tilde{k}\varepsilon^{1/4} X(t), \qquad k^* := \frac{k}{1-k}$$

In particular, for $\varepsilon^* = \tilde{k}^{4/3}$ the actual transaction costs (in terms of percentage of the traded volume) are ***always smaller*** than the transaction costs in the Morton-Pliska model.

ii) The ***actually achieved*** (asymptotical) growth rate for $\varepsilon = \varepsilon^*$ is ***bounded below*** by

$$r^* = r + \tfrac{1}{2}\tilde{\pi}'\sigma\sigma'\tilde{\pi} - 2\sqrt{\varepsilon^*}Tr(HM) = \tilde{R} - 2\sqrt{\varepsilon^*}Tr(HM).$$

In particular, stock investment under proportional transaction costs only yields a better performance (in terms of growth rate) than pure bond investment if we have

$$k^* < \left(\frac{1}{4}\frac{\tilde{\pi}'\sigma\sigma'\tilde{\pi}}{Tr(HM)}\right)^{3/4} / \|y^*\|_1 \; .$$

Consequence:

A practitioner's guide to applying the Morton-Pliska approach in reality:

1. determine M out of the market coefficients by solving the matrix equation (ME)
2. check if k is sufficiently small such that following the Morton-Pliska strategy yields a higher growth rate than pure bond investment
3. if the above check is positive then follow the Morton-Pliska strategy with $\varepsilon = \varepsilon^*$ as defined in Proposition i)

Example: 1stock, 1 bond, BS-market with

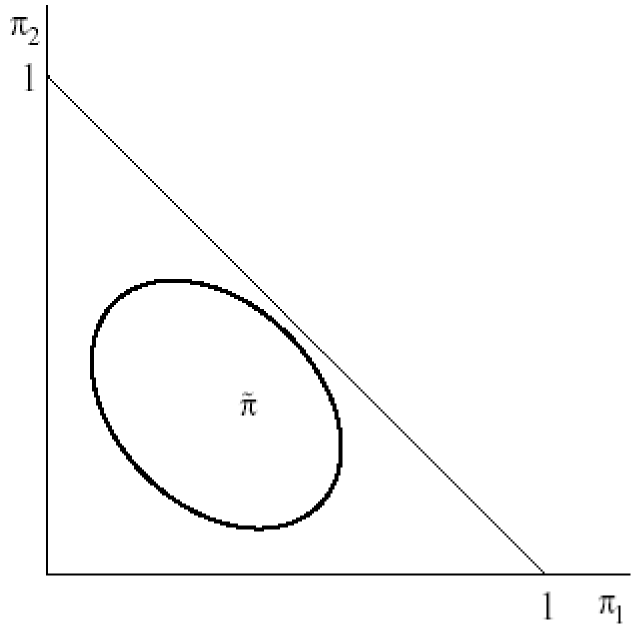
r = 0.05, b= 0.08, σ = 0.3 $\Rightarrow$ $\bar{\pi} = 1/3$, $\tilde{R} = 0.055$

k	0.001	0.01	0.05	0.1
Growth rate	0.0549	0.0545	0.0535	0.0526
Interval length (1/2)	0.084	0.182	0.230	**0.410**

Break-even transaction costs: k= 0.245146 (however, asymptotic expansion is doubtful !)

Example 2: Two stocks, one bond in the BS-model with

$r = 0.05$, $b_1 = 0.08$, $\sigma_{11} = 0.3$, $b_2 = 0.08$, $\sigma_{22} = 0.3$, $\sigma_{12} = 0 \Rightarrow \bar{\pi} = \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}$, $\bar{R} = 0.06$



k	0.001	0.01	0.5	0.1
Growth rate	0.0598	0.0589	0.0568	0.0547
"Interval length"	0.16	0.35	0.62	**0.79**

Break-even transaction costs: $k^* = 0.223655$

Advantage for practical use:

- We obtained a reasonable strategy
- We obtained upper bounds for the transaction costs
- We obtained lower bounds for the performance
- We only have to solve a quadratic matrix equation

$\Rightarrow$ **We have solved a practical problem via adapting an existing solution approach**

Open questions:

- Modification for including a ***fixed cost component*** in the real world ?.
- Modification allowing for power functions $f(x) = \frac{1}{\gamma} x^{\gamma}, \quad \gamma < 1, \gamma \neq 0$?
- Does a higher order expansion increase the accuracy ? How about the exact solution ?
- How good is the lower bound for the actual growth rate ?

6. Optimal investment with Stochastic Interest Rates

So far:

Bonds are treated just as bank accounts

$\Rightarrow$ constant interest rates are totally inappropriate over a long time horizon

Traditional measures are focused on **static** measuring of risk and return:

yield	$\cong$	average return
duration	$\cong$	role as delta in option trading
convexity	$\cong$	role as gamma in option trading

Why not applying modern **dynamic** methods ?

$\Rightarrow$ set up a bond portfolio problem

$\Rightarrow$ maximize expected utility (K., Kraft (2000), Kraft (2002), …)

Solving a Bond Portfolio Problem via Stochastic Control

Model assumptions and price equations:

$$dr(t) = a(t)dt + b\,dW(t) \qquad \text{“}\textbf{short rate}\text{”}$$

$$dB(t) = B(t)r(t)dt \qquad \text{“}\textbf{money market account}\text{”}$$

$$dP(t,T_1) = P(t,T_1)\left[\underbrace{\left(r(t)+\zeta(t)\sigma(t)\right)}_{=:\,\mu(t)}dt + \sigma(t)dW(t)\right] \qquad \text{“}\boldsymbol{T_1}\textbf{-bond}\text{”}$$

(think of Ho-Lee or Vasicek model as examples, more complicated examples are possible !)

$\Rightarrow$ “wealth equation”

$$dX^{\pi}(t) = X^{\pi}(t)\Big(\big(r(t)+\pi(t)\zeta(t)\sigma(t)\big)dt + \pi(t)\sigma(t)dW(t)\Big), \quad X^{\pi}(0) = x$$

“**portfolio problem**”

$$\boxed{\max_{\pi(.)\in A(x)} E\left(U\left(X^{\pi}(T)\right)\right) \qquad \text{with } T < T_1}$$

$\Rightarrow$ **Corresponding HJB-Equation**

$$\boxed{\begin{aligned} 0 = \max_{|\pi| \le \delta} \Big\{ & v_t(t,x,r) + \tfrac{1}{2}\left(x^2\pi^2\sigma^2 v_{xx}(t,x,r) - 2x\pi b\sigma v_{xr}(t,x,r) + b^2 v_{rr}(t,x,r)\right) \\ & + x(r + \pi\zeta\sigma) v_x(t,x,r) + a v_r(t,x,r) \Big\} \end{aligned}}$$

<u>**Step 1:**</u> "Solve the maximisation problem in the HJB-Equation"

$$\pi^*(t) = -\frac{\zeta(t)}{\sigma(t)} \frac{v_x(t,x,r)}{x v_{xx}(t,x,r)} - \frac{b}{\sigma(t)} \frac{v_{x,r}(t,x,r)}{x v_{xx}(t,x,r)}$$

"**classical part**" "**correction term**"

<u>**Step 2:**</u> "Substitute the maxima into the HJB-Equation and solve the resulting pde"

…..some really complicated equation… .

Example: "Power utility and Ho-Lee-model"

$$U(x) \quad = \quad x^{\gamma}, \;\; 0 < \gamma < 1$$

$dr(t) \quad = (\tilde{a}(t) + b\zeta(t))dt + bdW(t)$ for some continuous, deterministic function $\zeta(t)$

$$dP(t,T_1) = P(t,T_1)[(r(t) - \zeta(t)b(T_1 - t))dt - b(T_1 - t)dW(t)]$$

Important: For portfolio optimization one has to use the dynamics under the ***subjective measure*** *P*, not (!) under the equivalent martingale measure Q !!!

$\Rightarrow$

Solution of the portfolio problem can be given explicitly as

$$v(t,x,r) \; = \; x^{\gamma}\exp\left(\tfrac{1}{1-\gamma}\left(H_1(t) - H_1(T)\right) + \gamma(T-t)r\right)$$

$$\pi^*(t) \quad = \frac{1}{1-\gamma}\frac{\zeta(t) + \gamma b(T-t)}{-b(T_1 - t)} = -\frac{1}{1-\gamma}\frac{\zeta(t)}{b(T_1 - t)} - \frac{\gamma}{1-\gamma}\frac{T-t}{T_1 - t}$$

Correction term results in an initially "smaller" risky position, but vanishes asymptotically !

Counterexample:

Example: "Power utility and Dothan model"

$$U(x) = x^{\gamma}, \quad 0 < \gamma < 1$$

$$dr(t) = r(t)[b\,dt + \sigma\,dW(t)]$$

$$\Rightarrow \quad E\left(U\left(X^0(T)\right)\right) = +\infty$$

Generalisations:

- Mixed stock and bond problems
- Problems including bond and stock options
- ….

7. More Problems

- ***Optimal investment with constraints***
  - on the final wealth (see K. (1997), …)
  - on the portfolio process (see Cvitanic and Karatzas (1993), Kramkov and Schachermayer (1998), …)
  - on risk measures (Karatzas (1999), Uryasev e.a. (1999), Sass, Wunderlich (2005),…)
- ***Optimal portfolios in non-diffusion settings*** (Benth e.a. (…), Kallsen (..))
- ***Non-utility based approaches***
  - Universal portfolios (Cover (1995), Jamshidian (1995), …)
  - Approximation of the growth-optimal portfolio (Platen (2006), …)
  - Value preserving portfolios (Hellwig (1987), Wiesemann (1995), K. (1997), ..)
- ***Estimation problems for the market coefficients***

$\Rightarrow$ **Still a lot to do !**