Stochastic Agent-Based Models in Economics and Finance:

An Introduction to Quantitative Modelling Concepts for Interacting Heterogeneous Agents

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Agent-Based Models in Economics and Finance: Schedule

Stochastic Models of Interacting Agents: Structure and Quantitative Modeling Concepts

a. The Master Equation Formalism: Stationary Solutions and Transient Behavior

b. Dynamics of Means and Higher Moments

c. Heterogeneous Beliefs and Asset Price Dynamics

d. Artificial Markets with Herding and Strategy Choice

e. Estimation of Stochastic Agent-Based Models

A Stochastic Framework
for Socio-Economic Interactions

- similar to a discrete choice with social interactions (Brock/Durlauf), but dynamic framework and consideration of finite populations (no thermodynamic limit)

- we might dispend with utility maximization and simply formulate dynamic behavioral models of the time development of agents' decisions or activities

- since we cannot fully capture agents' idiosyncratic behavioral elements we adopt a stochastic approach of agents' choices

Motivating examples

- adopters and non-adopters of a technology
- fundamentalists versus chartists, bears versus bulls in financial markets
- supporters of government and opposition
- followers and non-followers of fashion

A Stochastic Framework for Socio-Economic Interactions

We consider a population with a binary set of choices or opinions (denoted by "+" and "-"):

$$n_+ + n_- = 2N$$

The *socio-economic configuration* at any point in time can be characterized by:

$$n = \frac{1}{2}(n_+ - n_-)$$

or the *opinion index* x:

$$x = \frac{n}{N}, \qquad x \in [-1,1]$$

Note that $n_+ = N + n$, $n_- = N - n$.

- ✓ We assume that the dynamics can be captured by certain *transition probabilities* for agents to move from the "+" to the "-"group and vice versa, p_{-+} and p_{+-} .
- ✓ This means that the population composition follows a stochastic process (which might have systematic components which enter via the specifications of p_{-+} and p_{+-}).

A complete characterization of the process requires to solve for the probability distribution at any point in time, *t*, over all possible states, *n* or *x:*

$$P(n;t) \quad with \quad \sum_{n=-N}^{N} P(n;t) = 1$$

$$or$$

$$P(x;t) \quad with \quad \sum_{x=-1}^{1} P(x;t) = 1$$

These time-dependent distributions might or might not converge to a *stationary distribution* for $t \rightarrow \infty$.

Dynamics: the probabilities to find the system in states n (or x) change over time according to the probabilities for movements of single individuals. For example, the configuration might change from n to $n+1$ or $n-1$ with one agent moving to another group with probabilities:

$$w(n \to n+1) = n_- p_{+-}(n;t) = (N-n) p_{+-}(n)$$

$$w(n \to n-1) = n_+ p_{-+}(n;t) = (N+n) p_{-+}(n)$$

assuming that there is an influence of the overall configuration n on individuals' propensities to move between groups and that it is only the overall number of agents in the "+" and "-" groups, that influences these decisions.

Transitions in Continuous Time: Jump Markov Processes

We assume that time is continuous and that individual switches can be formalized via *asynchronous Poisson processes in continuous time*. We can, then, specify the dynamic process more formally via conditional probabilities, e.g. $\omega(n+1,t+\Delta t|n,t)$, $\omega(n-1,t+\Delta t|n,t)$, etc. and in the limit $\Delta t \rightarrow 0$ we define:

$$\lim_{\Delta t \to 0} \frac{\omega(n+1, t+\Delta t | n, t)}{\Delta t} = w(n+1|n,t) = w_{\uparrow}(n),$$

$$\lim_{\Delta t \to 0} \frac{\omega(n-1, t+\Delta t | n, t)}{\Delta t} = w(n-1|n,t) = w_{\downarrow}(n).$$

For $\Delta t \rightarrow 0$ two or more simultaneous movements of individuals become increasingly unlikely and the probability for one individual to change his mind converges to $\lambda \Delta t$ with λ the *transition rate* of the Poisson process. The Poisson transition rates for the population process are, therefore, given by:

$$w(n+1|n,t) \equiv w_{\uparrow}(n)=n_{-}p_{+-} \qquad \text{and}$$

$$w(n-1|n,t) \equiv w_{\downarrow}(n)=n_{+}p_{-+}.$$

Reminder: the Poisson distribution counts the number of events during a time interval. For a Poisson rate λ we get:

$$P_n(\Delta t) = \frac{(\lambda \Delta t)^n}{n!} e^{-\lambda \Delta t}$$

$$P_1(\Delta t) = \lambda \Delta t \cdot e^{-\lambda \Delta t} = \lambda \Delta t (1 - \lambda \Delta t + \frac{(\lambda \Delta t)^2}{2} + \ldots)$$

$$P_2(\Delta t) = \frac{(\lambda \Delta t)^2}{2} e^{-\lambda \Delta t} = o(\Delta t)$$

For $\Delta t \to 0$, only the first term of $P_1(\Delta t)$ needs to be considered since simultaneous movements of agents become increasingly unlikely. *Note, however, that our 'λ' might be state-dependent and nonlinear!*

The overall evolution of the system is characterized by the time change of the probabilities over all states.

In general, this amounts to a system of difference equations for all possible system configurations *n* (the Master equation):

$$P(n,t+\Delta t)-P(n,t)=$$

$$\underbrace{\sum_{n'}\omega\left(n,t+\Delta t\middle|n',t\right)P\left(n';t\right)}_{\text{inf low of probability to state } n}-\underbrace{\sum_{n'}\omega\left(n',t+\Delta t\middle|n,t\right)P\left(n;t\right)}_{\text{outflow of probability from state } n}$$

Master Equation in Continuous Time

In the continuous-time limit we get the vector differential equation:

$$\frac{dP(n;t)}{dt} = \sum_{n'} \{ w(n|n',t)P(n';t) - w(n'|n,t)P(n,t) \}$$

for the probability flux. Note that in the continuous-time limit, transition probabilities *ω(.)* have been replaced by transition rates *w(.)* on the right-hand side of the Master equation.

In our case of Poisson jump processes of single agents, we can restrict attention to neighboring states, *n' = n ± 1*:

$$\frac{dP(n;t)}{dt} = w_{\downarrow}(n+1)P(n+1;t) + w_{\uparrow}(n-1)P(n-1;t)$$

$$-\left(w_{\uparrow}(n) + w_{\downarrow}(n)\right)P(n;t)$$

Assuming that the transition probabilities of individuals do not depend on the raw numbers, but rather on the ratio of members of both groups, we can also express the dynamics in terms of the opinion index *x*.

Since $\Delta n = 1$ corresponds to $\Delta x = 1/N$, we get:

$$w_{\uparrow}(x) = w\left(x + \frac{1}{N} \middle| x, t\right) = n_{-} p_{+-}(x) = N(1 - x) p_{+-}(x)$$

$$w_{\downarrow}(x) = w\left(x - \frac{1}{N} \middle| x, t\right) = n_{+} p_{-+}(x) = N(1 + x) p_{-+}(x)$$

Master equation for opinion index x:

$$\frac{dP(x;t)}{dt} = w_{\downarrow}\left(x+\frac{1}{N}\right)P\left(x+\frac{1}{N};t\right) + w_{\uparrow}\left(x-\frac{1}{N}\right)P\left(x-\frac{1}{N};t\right)$$
$$-\left(w_{\uparrow}(x)+w_{\downarrow}(x)\right)P(x;t)$$

The Fokker-Planck Equation as a Taylor Series Expansion of the Master Equation

Since for large *N*, *x* is close to a continuous quantity, we can perform a Taylor expansion with respect to Δx. Rearranging gives:

$$\frac{dP(x;t)}{dt} = w_{\uparrow}\left(x - \frac{1}{N}\right)P\left(x - \frac{1}{N};t\right) - w_{\uparrow}(x)P(x;t)$$

$$+ w_{\downarrow}\left(x + \frac{1}{N}\right)P\left(x + \frac{1}{N};t\right) - w_{\downarrow}(x)P(x;t)$$

A second-order approximation of the first and second group of components on the right-hand side around *x* yields:

$$\frac{\partial P(x;t)}{\partial t} = \frac{\partial}{\partial x}[w_{\uparrow}(x)P(x;t)]\left(-\frac{1}{N}\right) + \frac{1}{2}\frac{\partial}{\partial x^2}[w_{\uparrow}(x)P(x;t)]\left(-\frac{1}{N}\right)^2 + \frac{\partial}{\partial x}[w_{\downarrow}(x)P(x;t)]\frac{1}{N} + \frac{1}{2}\frac{\partial}{\partial x^2}[w_{\downarrow}(x)P(x;t)]\left(\frac{1}{N}\right)^2$$

$$\Rightarrow \frac{\partial P(x;t)}{\partial t} = -\frac{\partial}{\partial x}[(w_{\uparrow}(x) - w_{\downarrow}(x))P(x;t)]\frac{1}{N} + \frac{1}{2}\frac{\partial}{\partial x^2}[(w_{\uparrow}(x) + w_{\downarrow}(x))P(x;t)]\frac{1}{N^2}$$

Fokker-Planck equation

The Fokker-Planck Equation

The FP equation consists of two terms:

- the drift term for the systematic (mean-value) part of the dynamics

$$A(x) = \frac{1}{N}\left(w_{\uparrow}(x) - w_{\downarrow}(x)\right)$$

- the diffusion term for the fluctuations around the expected value:

$$D(x) = \frac{1}{N^2}\left(w_{\uparrow}(x) + w_{\downarrow}(x)\right)$$

Remarks

- the Fokker-Planck equation maintains only terms up to second order, so that it amounts to a Gaussian approximation of the 'true' probability distribution at time t (nevertheless, the time evolution leads to different shapes of the transient and stationary distributions),
- for simple drift and diffusion functions, it might be possible to solve explicitly for the stationary distribution from $dP(x;t)/dt = 0$,
- in general, the Fokker-Planck equation can be analyzed and simulated more easily than the master equation,
- the N^2 dependence of the diffusion term in our setting is a consequence of the law of large numbers.

Derivation of Macroscopic Laws of Motion from the Microdynamics

The expected value of *x* at time *t* is given by:

$$\overline{x}_t = \sum_{x=-1}^{1} xP(x;t)$$

and its change in time is obtained from the Master equation:

$$\frac{d\overline{x}_t}{dt} = \sum_{x=-1}^{1} x \frac{dP(x;t)}{dt}$$

In general:

$$\frac{d\overline{x}_t}{dt} = \sum_{x} x \sum_{x'} (w(x|x',t)P(x',t) - w(x'|x,t)P(x;t))$$

With a few simple manipulations we get:

$$\frac{d\overline{x}_t}{dt} = \sum_x \sum_{x'} x w_{xx'} P(x';t) - \sum_x \sum_{x'} x w_{x'x} P(x;t)$$

$$= \sum_x \sum_{x'} x' w_{x'x} P(x;t)) - \sum_x \sum_{x'} x w_{x'x} P(x;t)$$

$$= \sum_x \underbrace{\sum_{x'} (x'-x) w_{x'x}}_{\equiv a_{x,1}} P(x,t)$$

$$= \sum_x a_{x,1} P(x;t) = \overline{a_{x,1}}$$

$$\text{with } w_{x'x} = w(x' \mid x,t)$$

$a_{x,1}$ is the so-called *first jump moment* which, in our framework, coincides with the drift term of the Fokker Planck equation:

$$a_{x,1} = \sum_{x'} (x' - x) w_{x'x} = \frac{1}{N} w_{\uparrow}(x) + \left(-\frac{1}{N}\right) w_{\downarrow}(x)$$

$$= \frac{1}{N} (w_{\uparrow}(x) - w_{\downarrow}(x))$$

The change in time of the expected value of x is, therefore, given by the expectation of $a_{x,1}$, i.e., the average jump of x weighted by the probability of x at time t, $P(x;t)$:

$$\frac{d\overline{x}_t}{dt} = \sum_x a_{x,1} P(x;t) = \overline{a_{x,1}}$$

Approximation of mean-value equation:

First-order approximation of the mean-value dynamics yields:

$$\frac{d\bar{x}_t}{dt} = a_{x,1}(\bar{x})$$

which is exact if the first jump moment is linear in x.

A *second-order approximation* adds a correction term depending on the fluctuations:

$$\frac{d\bar{x}_t}{dt} = E[\,a_{x,1}(\bar{x}) + \underbrace{(x-\bar{x})}_{=0} a'_{x,1}(\bar{x}) + \frac{1}{2}\underbrace{(x-\bar{x})^2}_{\sigma_x^2} a''_{x,1}(\bar{x}) + \ldots\,]$$

$$= \underbrace{a_{x,1}(\bar{x})}_{\text{pure mean value dynamics}} + \underbrace{\frac{1}{2}\sigma_x^2 a''_{x,1}(\bar{x})}_{\text{second-order correction}}$$

Simple example: birth/death dynamics

$$w(n+1|n,t) = \lambda n \equiv r_n, w(n-1|n,t) = \mu(\frac{n}{N})n \equiv l_n$$

With λ, μ: birth and death rates, N: maximum population, carrying capacity of environment

Master equation:

$$\frac{dP(n;t)}{dt} = r_{n-1}P(n-1;t) + l_{n+1}P(n+1;t) - (r_n + l_n)P(n;t)$$

First jump moment:

$$a_{n,1} = \sum_{n'} (n'-n)w(n \to n') = 1 \cdot r_n + (-1) \cdot l_n = \lambda n - \mu \frac{n^2}{N}$$

Mean values in first-order approximation:

$$\frac{d\,\overline{n_t}}{dt} = \overline{a_{n,1}} = \overline{\lambda n - \mu \frac{n^2}{N}} \cong \lambda \bar{n} - \mu \frac{\bar{n}^2}{N}$$

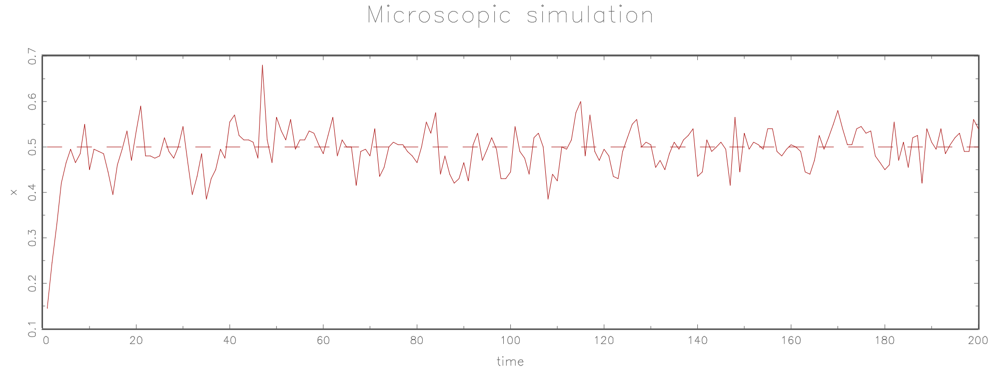
In concentrations, x = n/N, we get:

$$\frac{dP(x;t)}{dt} = r_{x-\frac{1}{N}} P\left(x - \frac{1}{N}; t\right) + l_{x+\frac{1}{N}} P\left(x + \frac{1}{N}; t\right) - (r_x + l_x)P(x;t)$$

$$\frac{d\,\overline{x_t}}{dt} = \overline{a_{x,1}} = \overline{\lambda x - \mu x^2} \cong \lambda \bar{x} - \mu \bar{x}^2$$

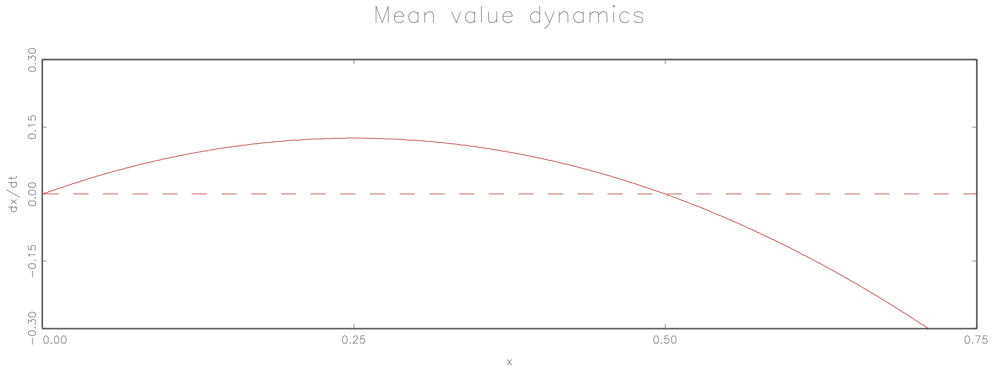
Second-order approximation:

$$a'_{x,1} = \lambda - 2\mu x;\ a''_{x,1} = -2\mu$$



Leads to:

$$\frac{d\overline{x_t}}{dt} \cong \lambda\overline{x} - \mu\overline{x}^2 - \mu\sigma_x^2$$



Equilibria:

$$\frac{d\overline{x_t}}{dt} = 0 \Leftrightarrow \overline{x^*} = \frac{\lambda}{\mu} \quad \left(\overline{n^*} = \frac{\lambda}{\mu}N\right)$$

λ=1, μ = 2, N = 200

2nd Example: a pure social dynamics or herding process

$$p_{+-} = ve^{\alpha_0 + \alpha_1 x}, p_{-+} = ve^{-\alpha_0 - \alpha_1 x}$$

With v: frequency of jumps, α_0: bias, α_1: herding intensity.

Inspiration:

Discrete choice (in economics):

$$prob(buy\ brand\ A) = \frac{e^{\beta U_A}}{e^{\beta U_A} + e^{\beta U_B} + \ldots}$$

Particle dynamics (in physics): magnetization etc.

The population transition rates are:

$$w_{\uparrow}(x) = \underbrace{N(1-x)}_{n_-}ve^{\alpha_0+\alpha_1 x}$$

$$w_{\downarrow}(x) = \underbrace{N(1+x)}_{n_+}ve^{-\alpha_0-\alpha_1 x}$$

The first jump moment is:

$$\begin{aligned} a_{x,1} &= \sum_{x'}(x'-x)w_{x'x} \\ &= \frac{1}{N}[N(1-x)ve^{\alpha_0+\alpha_1 x} - N(1+x)ve^{-\alpha_0-\alpha_1 x}] \end{aligned}$$

Using the hyperbolic trigonometric functions, the first-order approximation to the mean-value dynamics becomes:

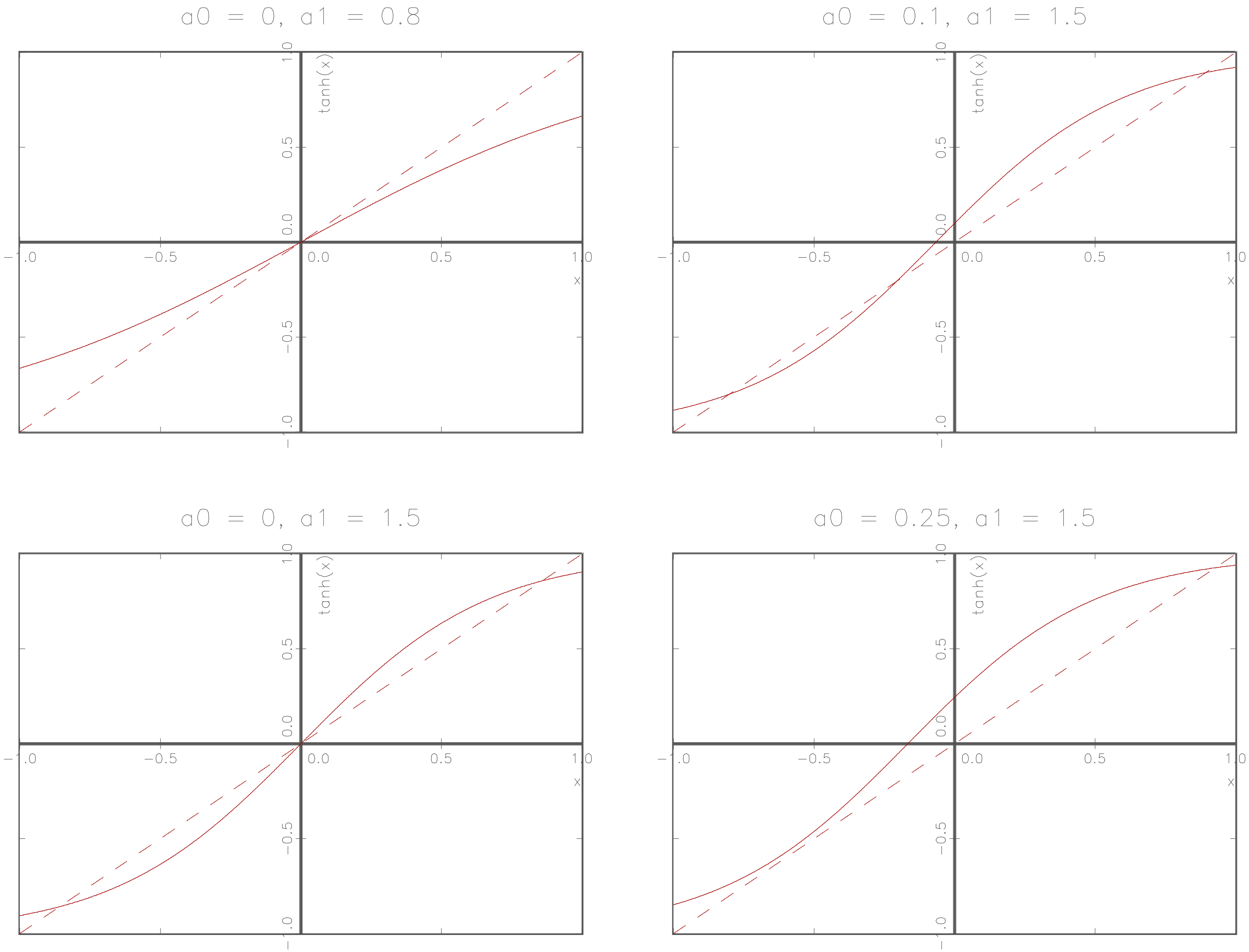
$$\frac{d\bar{x}_t}{dt} = 2v(Tanh(\alpha_0 + \alpha_1\bar{x}) - \bar{x})Cosh(\alpha_0 + \alpha_1\bar{x})$$

Stationary states are obtained from:

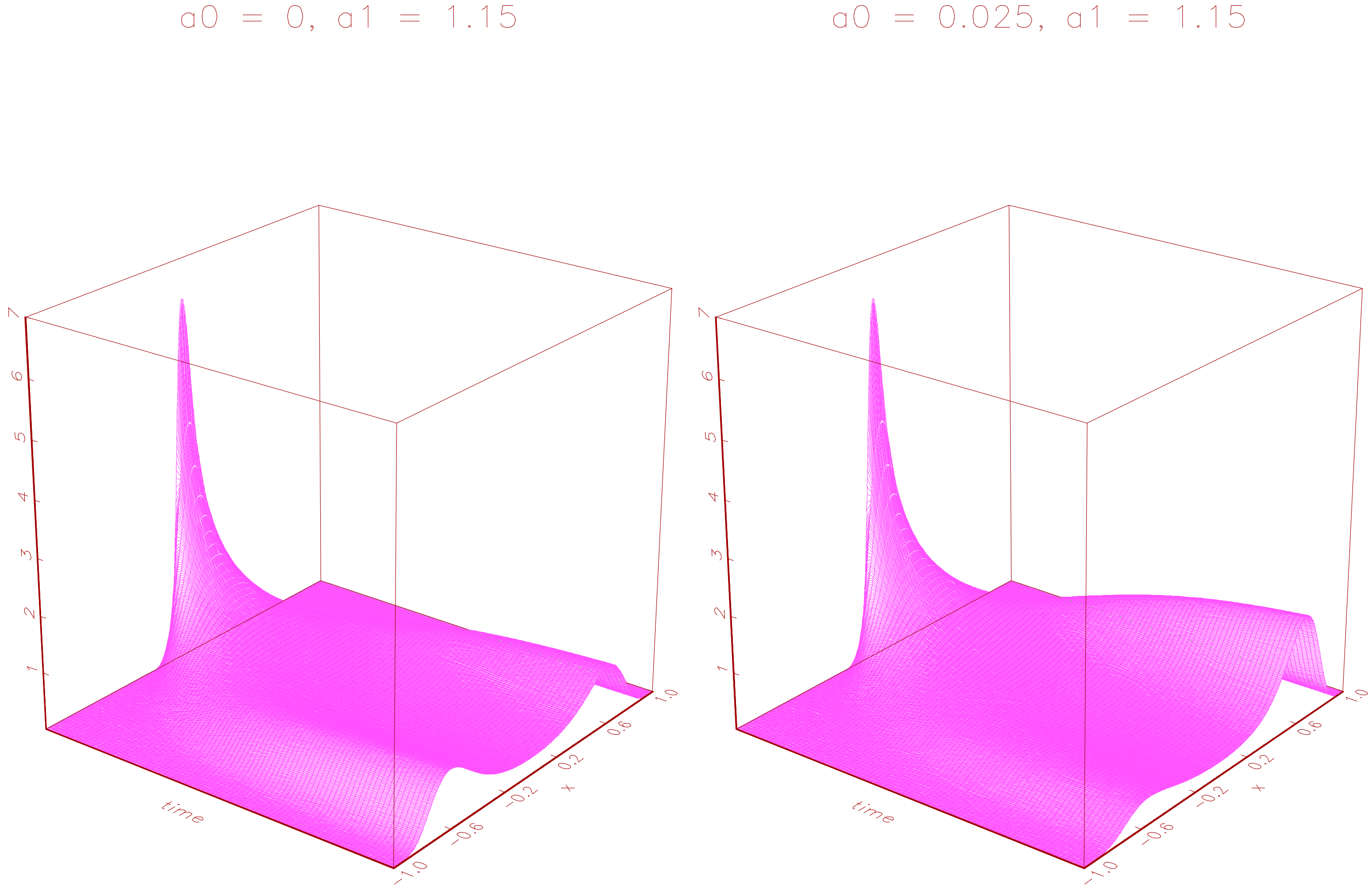
$$\frac{d\bar{x}_t}{dt} = 0 \Rightarrow \bar{x}^* = Tanh(\alpha_0 + \alpha_1\bar{x}^*)$$

Features:

- unique stable steady state x* for α_0 small, $\alpha_1 \leq 1$,
- multiple steady states $x_{\pm}^* \neq 0$ for α_0 small, $\alpha_1 > 1$, the formerly unique steady state x* becomes unstable.



Various cases of the stochastic herding model



Examples of Transient Densities from Fokker-Planck Equation

Second-order approximation:

$$\frac{d\overline{x}_t}{dt} = 2v(Sinh(\alpha_0 + \alpha_1\overline{x}) - \overline{x}Cosh(\alpha_0 + \alpha_1\overline{x})) \;+\; \underbrace{v((\alpha_1{}^2 - 2\alpha_1)Sinh(\alpha_0 + \alpha_1\overline{x}) - \overline{x}\alpha_1 Cosh(\alpha_0 + \alpha_1\overline{x}))}_{\frac{1}{2}a''_{x,1}(\overline{x})}\sigma_x^2$$

To implement the correction term, we need information about the time evolution of the variance!

Dynamics of Second Moments

Similarly as with first moments, we see that the second moment is time dependent and its evolution can be obtained via the Master equation:

$$\overline{x_t^2} = \sum_x x^2 P(x;t)$$

$$\begin{aligned}
\frac{d}{dt}\overline{x_t^2} &= \sum_x x^2 \frac{dP(x;t)}{dt} \\
&= \sum_x x^2 \sum_{x'} (w_{xx'} P(x';t) - w_{xx'} P(x;t)) \\
&= \sum_x \sum_{x'} (x'^2 - x^2) w_{x'x} P(x;t) \\
&= \sum_x \sum_{x'} ((x'-x)^2 + 2x(x'-x)) w_{x'x} P(x;t)
\end{aligned}$$

Defining the *second jump moment*:

$$\sum_{x'} (\mathrm{x'\text{-}x})^2 w_{x'x} = a_{x,2}$$

We arrive at the ***exact*** dynamics of the second moment of *x*:

$$\frac{d\overline{x_t^2}}{dt} = \sum_x (a_{x,2} + 2xa_{x,1})P(x;t)$$
$$= \overline{a_{x,2}} + 2\overline{xa_{x,1}}$$

The dynamics of the variance is obtained as:

$$\frac{d}{dt}\sigma_x^2 = \frac{d}{dt}\left(\overline{x^2} - \bar{x}^2\right) = \frac{d}{dt}\overline{x^2} - \frac{d}{dt}\bar{x}^2$$

$$= \overline{a_{x,2}} + 2\overline{x a_{x,1}} - 2\bar{x}\,\overline{a_{x,1}}$$

$$= \overline{a_{x,2}} + 2\overline{\left(x - \bar{x}\right)a_{x,1}}$$

Again we apply a Taylor series expansion around the current expectation:

$$\frac{d}{dt}\sigma_x^2 = E[\,a_{x,2}(\bar{x}) + \underbrace{(x-\bar{x})}_{=0} a'_{x,2}(\bar{x})$$

$$+ \underbrace{2(x-\bar{x})}_{=0} a_{x,1}(\bar{x}) + 2\underbrace{(x-\bar{x})^2}_{\sigma_x^2} a'_{x,1}(\bar{x}) + \ldots]$$

$$\approx a_{x,2}(\bar{x}) + 2\sigma_x^2 a'_{x,1}(\bar{x})$$

If both $a_{x,1}$ and $a_{x,2}$ are linear, the last line is again exact, if not, it is an approximation up to the *first order*.

1st example: birth-death process

$$a_{x,2} = \sum_{x'} \left(x' - x\right)^2 w_{x'x} = \frac{1}{N^2}\left(\lambda n + \mu \frac{n}{N} n\right) = \frac{1}{N}\left(\lambda x + \mu x^2\right)$$

We arrive at a simultaneous system of equations:

$$\frac{d}{dt}\bar{x} = \lambda \bar{x} - \mu \bar{x}^2 - \mu \sigma_x^2$$

$$\frac{d}{dt}\sigma_x^2 = \frac{1}{N}\left(\lambda \bar{x} + \mu \bar{x}^2\right) + 2\sigma_x^2(\lambda - 2\mu\bar{x})$$

with or without: first vs. second order!

Variance in equilibrium for first-order approximation:

$$\overline{x}^* = \frac{\lambda}{\mu}, \sigma_x^2(\overline{x}^*) = \frac{-(\lambda \overline{x} + \mu \overline{x}^2)}{2N(\lambda - 2\mu \overline{x})} = \frac{\lambda}{N\mu}$$

… gets more complicated in simultaneous solution, but for large N is numerically not too different:

Example: $\lambda = 1$, $\mu = 2$

	x* (first order)	σ_x^2	x* (second order)	σ_x^2
N=200	0.5	0.0025	0.4949	0.002513
N=20	0.5	0.025	0.4375	0.02734

2nd example: herding model

$$a_{x,2} = \sum_{x'} \left(x' - x\right)^2 w_{x'x} = \frac{1}{N^2}\left(w_\uparrow(x) + w_\downarrow(x)\right)$$

which is the diffusion term *D(x)* from the Fokker-Planck equation.

We arrive at (for α_0=0, $\alpha_1 = \alpha$):

$$\frac{d}{dt}\sigma_x^2 = \underbrace{\frac{2\nu}{N}\left(Cosh\left(\alpha\bar{x}\right) - \bar{x}Sinh\left(\alpha\bar{x}\right)\right)}_{a_{x,2}(\bar{x})}$$

$$= \underbrace{4\nu\left((\alpha - 1)Cosh\left(\alpha\bar{x}\right) - \bar{x}\alpha Sinh\left(\alpha\bar{x}\right)\right)}_{2a'_{x,1}(\bar{x})}\sigma_x^2$$

The variance in a steady state, x*, is obtained via:

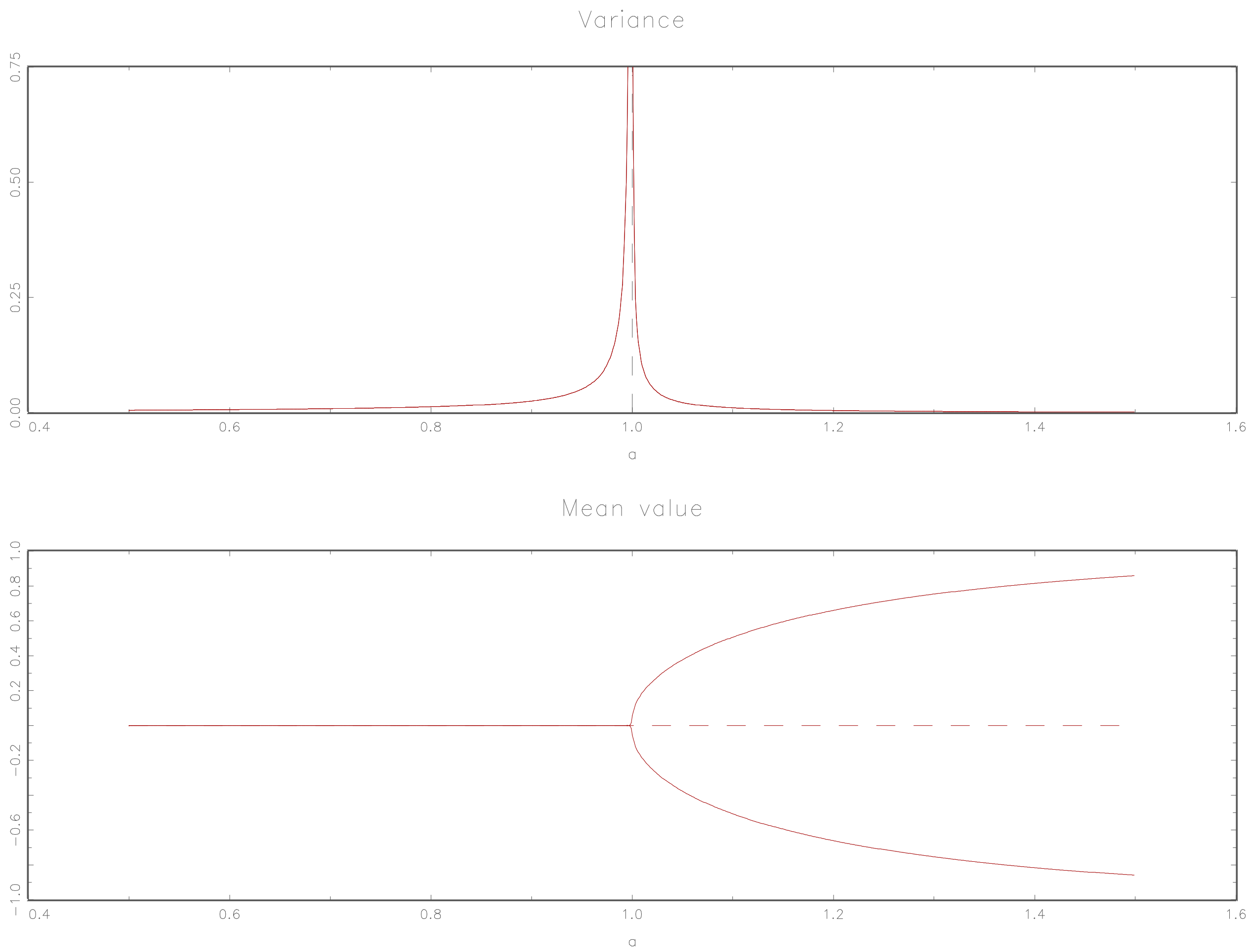
$$\frac{d}{dt}\sigma_x^2 = 0$$

$$\Rightarrow \overline{\sigma_x^2} = \frac{-\left(Cosh\left(\alpha\,\bar{x}\right) - \bar{x}Sinh\left(\alpha\,\bar{x}\right)\right)}{2N\left((\alpha - 1)Cosh\left(\alpha\,\bar{x}\right) - \alpha\,\bar{x}Sinh\left(\alpha\,\bar{x}\right)\right)}$$

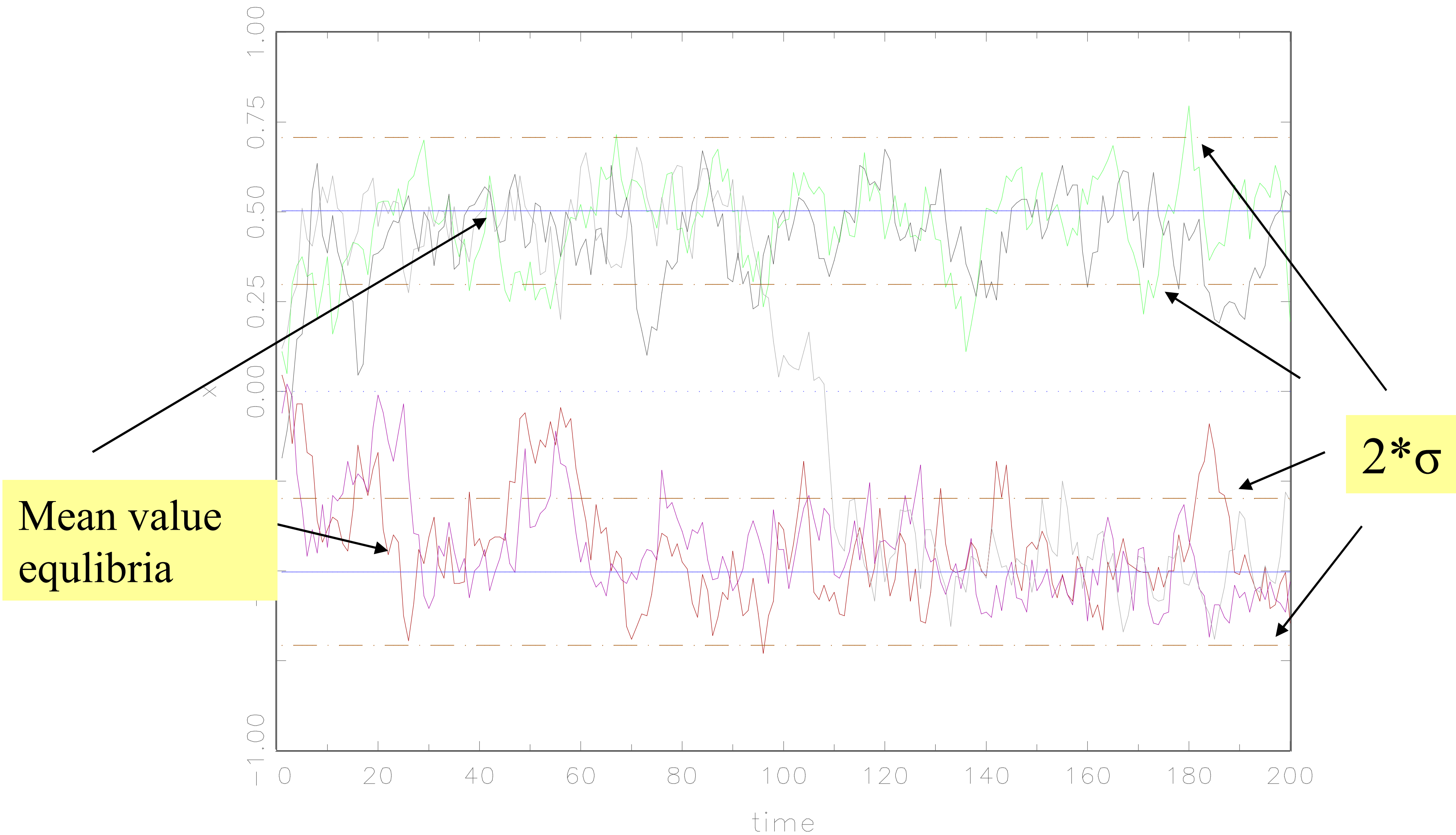
We get:

$$\overline{\sigma_x^2}\left(\bar{x} = 0\right) = \frac{1}{2N(1-\alpha)}$$

$$\overline{\sigma_x^2}\left(\overline{x_\pm^{\;*}}\right) = \frac{1}{2N\left(Cosh^2\left(\alpha\,\overline{x_\pm^{\;*}}\right) - \alpha\right)}$$

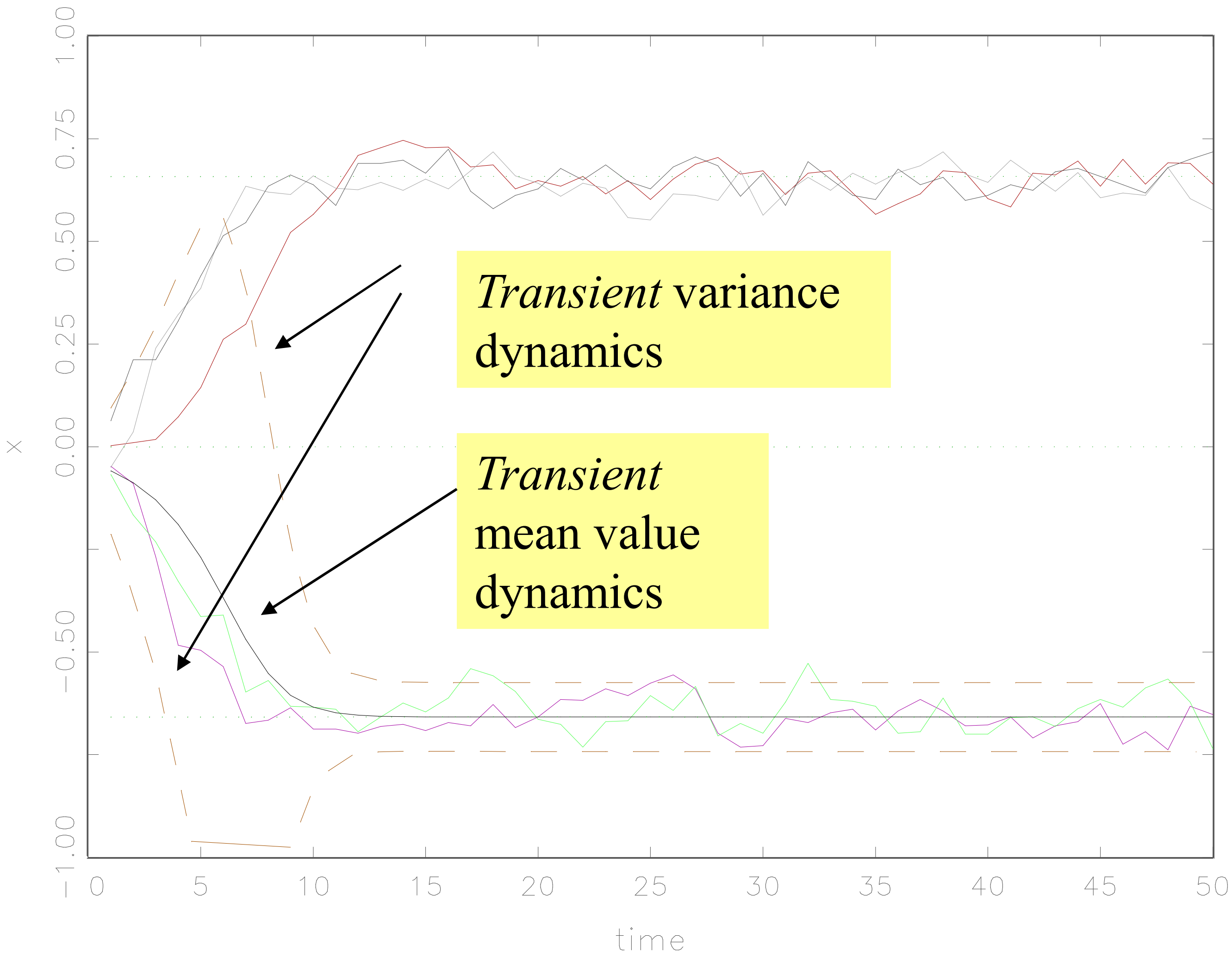


Mean value equilibria and stationary variances for varying α_1 ($\alpha_0 = 0$)



5 samples of herding model: N = 200; $\alpha_0 = 0$; $\alpha_1 = 1.1$; v = 1

Remark: variance is *local*, covers only fluctuations within basin of attraction



5 samples of herding model: N = 500; $\alpha_0 = 0$; $\alpha_1 = 1.2$; v = 1,
Initial condition: $x_0 = -0.04$

A Financial Market Model

The '+' and '-' groups are now identified as *bullish* and *bearish* speculators.

Transition probabilities account for:

- bias
- herding
- reinforcement of herding by momentum

$$p_{+-} = ve^{\alpha_0+\alpha_1 x+\alpha_2 \frac{p'(t)}{v}}, p_{-+} = ve^{-\alpha_0-\alpha_1 x-\alpha_2 \frac{p'(t)}{v}}$$

Note: *1/v* is mean time between changes of opinion

Dynamics of sentiment index *x*:

$$\frac{d\bar{x}_t}{dt} = 2v(Sinh(\alpha_0 + \alpha_1\bar{x} + \alpha_2\frac{p'(t)}{v}) - \bar{x}Cosh(\alpha_0 + \alpha_1\bar{x} + \alpha_2\frac{p'(t)}{v}))$$

Price dynamics from chartist-fundamentalist literature:

$$\frac{d}{dt}p = \beta ED = \beta(ED_c + ED_f)$$

β: price adjustment speed, ED: excess demand

Chartists are the bullish/bearish agents:

$$ED_c = (n_+ - n_-)t_c = 2Nxt_c = xT_c, T_c \equiv 2Nt_c$$

t_c: fixed trading volume per agents.

Fundamentalists have standard excess demand function:

$$ED_f = (p_f - p_t)T_f$$

Dynamic system (in mean values to first-order approximation)

$$\frac{d\bar{x}}{dt} = 2v(Tanh(U) - \bar{x})Cosh(U) \quad ,U = \alpha_0 + \alpha_1\bar{x} + \alpha_2\frac{p'(t)}{v}$$

$$\frac{dp}{dt} = \beta(T_c x + T_f(p_f - p))$$

Dynamic equilibrium: $$\frac{d\bar{x}_t}{dt} = \frac{dp}{dt} = 0$$

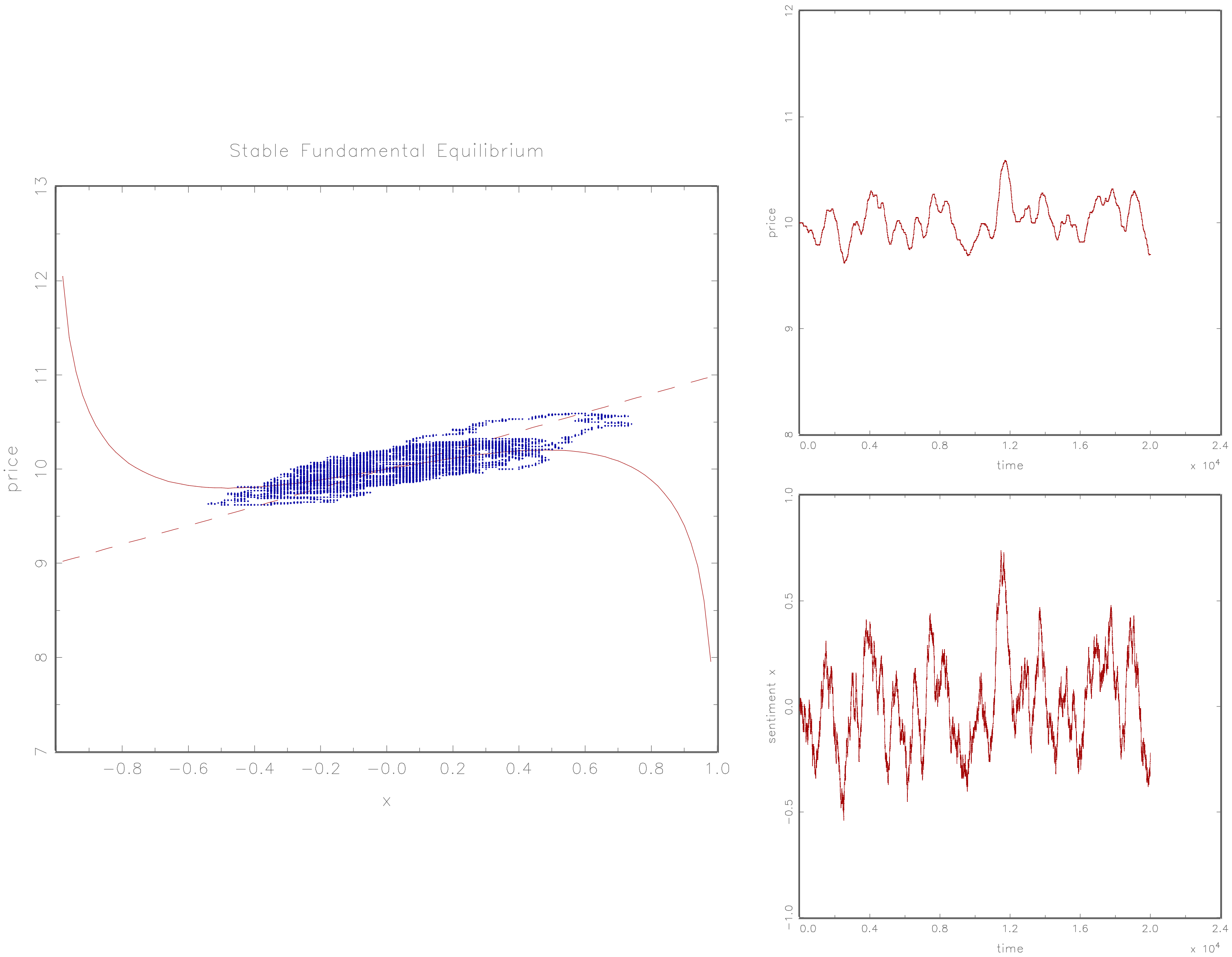
Stationary states are obtained from:

$$\frac{d\bar{x}_t}{dt} = 0 \Rightarrow \bar{x}^* = Tanh(\alpha_0 + \alpha_1 \bar{x}^*), p^* = \frac{T_c}{T_f} \bar{x}^* + p_f$$

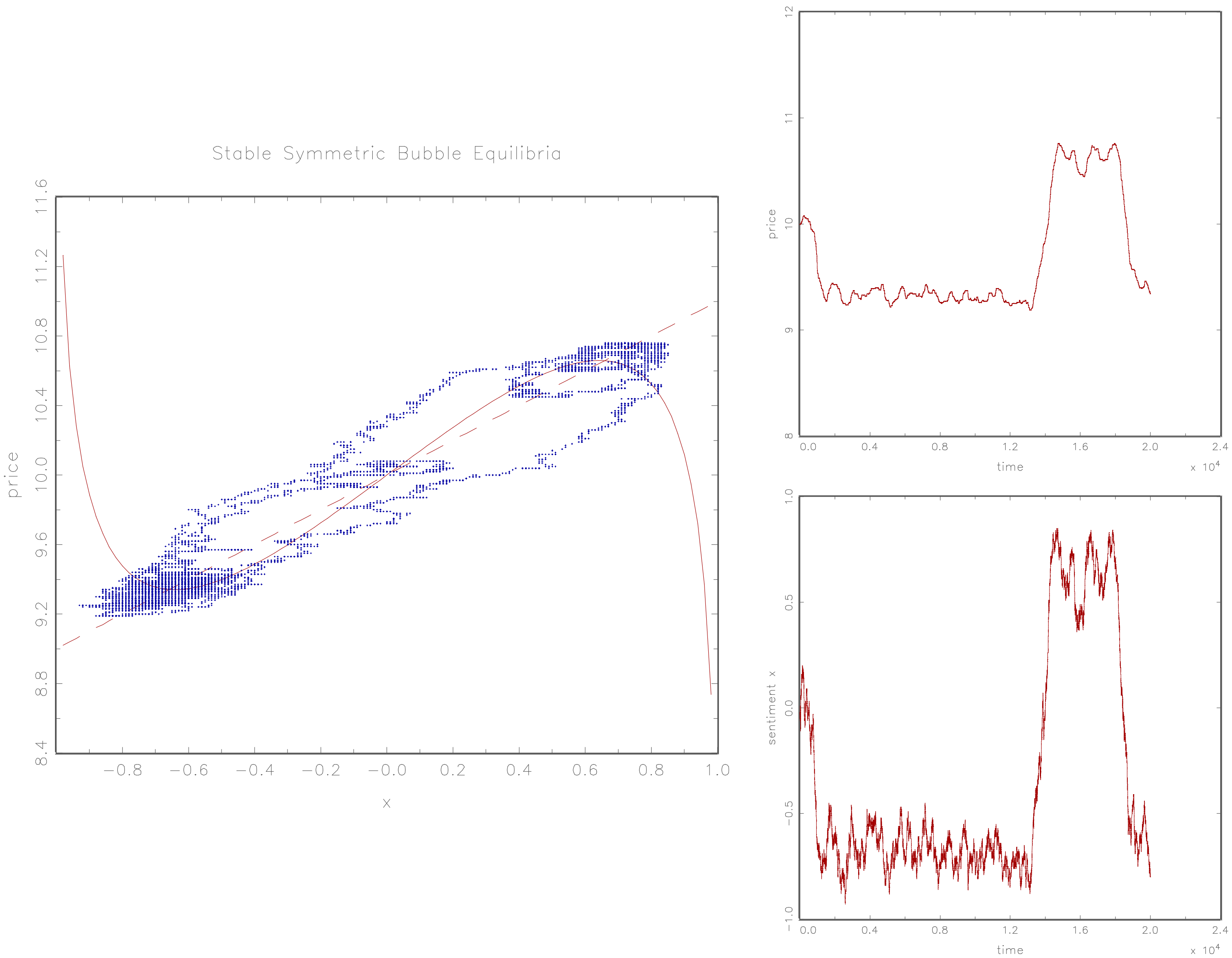
Features:

• unique steady state x* = 0 for α_0 small, $\alpha_1 \leq 1$, *together with price equal to fundamental value*

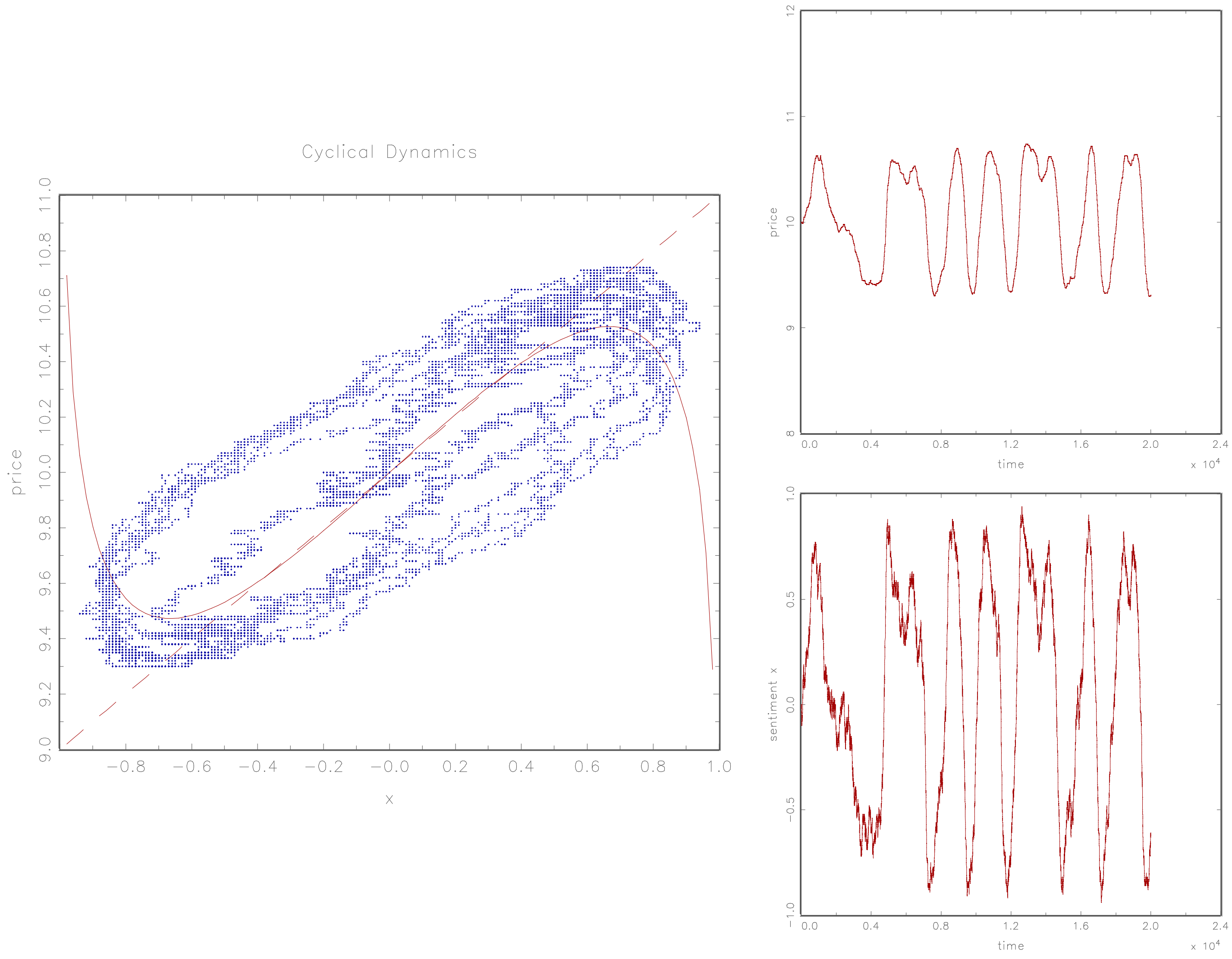
• multiple steady states for α_0 small, $\alpha_1 > 1$, two symmetric stable staedy states $x_{\pm}^* \neq 0$ for $\alpha_0 = 0$, *together with steady state prices deviating from fundamental value (bubble equilibria)*



Stable Fundamental Equilibrium
price
x
−0.8 −0.6 −0.4 −0.2 −0.0 0.2 0.4 0.6 0.8 1.0
7 8 9 10 11 12 13
price
time
x 10^4
0.0 0.4 0.8 1.2 1.6 2.0 2.4
8 9 10 11 12
sentiment x
time
x 10^4
−1.0 −0.5 0.0 0.5 1.0



Stable Symmetric Bubble Equilibria
price
x
time
sentiment x
x 10^4


Cyclical Dynamics
price
x
9.0
9.2
9.4
9.6
9.8
10.0
10.2
10.4
10.6
10.8
11.0
−0.8
−0.6
−0.4
−0.2
−0.0
0.2
0.4
0.6
0.8
1.0
price
time
8
9
10
11
12
0.0
0.4
0.8
1.2
1.6
2.0
2.4
x 10^4
sentiment x
time
−1.0
−0.5
0.0
0.5
1.0
x 10^4

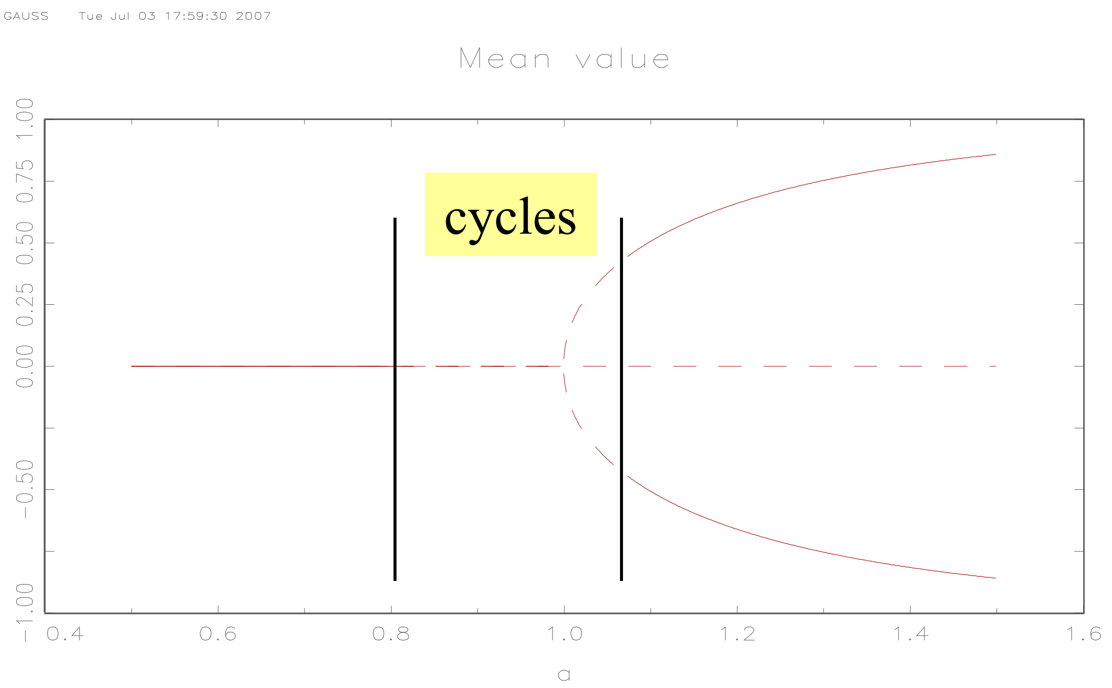
Stability conditions:

$$(i)\ for\ \ x^* = 0 : \alpha_1 \leq 1$$

$$\forall\ 2(v(\alpha_1 - 1) + \alpha_2 \beta T_c) - \beta T_f < 0$$

$$(ii)\ for\ x_{\pm}^* \neq 0 : \alpha_1 > 1$$

$$\forall\ 2\{(v\alpha_1 + \alpha_2 \beta T_c)(1 - x_{\pm}^*) - v\} Cosh(\alpha_1 x_{\pm}^*) - \beta T_f < 0$$



Beyond mean values: are there some interesting results in higher moments?

… the system we have formulated is a mixture of stochastic and deterministic components:

$$\frac{d\bar{x}}{dt} = 2v(Tanh(U) - \bar{x}Cosh(U)) \quad ,U = \alpha_0 + \alpha_1\bar{x} + \alpha_2\frac{p'(t)}{v}$$

$$\frac{dp}{dt} = \beta(T_c x + T_f(p_f - p))$$

Exact solution: bi-variate Master equation:

$$dQ(x,p;t)/dt = \text{inflow from other states - outflow from } \{x, p\}$$

$$= \left(w_{xx'}(t)\, Q(x',p;t) - w_{x'x}(t)\, Q(x,p;t)\right)$$

$$+ \left(w_{pp'}(t)\, Q(x,p';t) - w_{p'p}(t)\, Q(x,p;t)\right)$$

$w_{ii'}(t)$: transition rate in t for a transition from i' to i ($i = x,p$)

… where I assume that x and p do not move simultaneously (otherwise we would have to consider additional terms of format $w_{xx'pp'}(t)\, Q(x',p';t)$)

Mean value equations:

$$\bar{x} = \sum_x \sum_p x\, Q(x,p;t), \bar{p} = \sum_x \sum_p p\, Q(x,p;t)$$

$$\frac{d}{dt}\bar{x} = \sum_x \sum_p x \frac{d}{dt} Q(x,p;t), \frac{d}{dt}\bar{p} = \sum_x \sum_p p \frac{d}{dt} Q(x,p;t)$$

… leads to :

$$\begin{aligned}\frac{d\bar{x}_t}{dt} &= \sum_x \sum_p \sum_{x'} x w_{xx'} Q(x',p;t) - \sum_x \sum_{x'} x w_{x'x} Q(x,p;t) \\ &= \sum_x \sum_p \sum_{x'} x' w_{x'x} Q(x,p;t)) - \sum_x \sum_{x'} x w_{x'x} Q(x,p;t) \\ &= \sum_x \sum_p \underbrace{\sum_{x'} (x'-x) w_{x'x}}_{\equiv a_{x,1}} Q(x,p,t) \\ &= \sum_x \sum_p a_{x,1} Q(x,p;t) = \overline{a_{x,1}(x,p)}\end{aligned}$$

…same for p

Taylor series expansion:

$$\frac{d}{dt}\bar{x} = a_{x,1}(\bar{x},\bar{p}) + 0.5\{\sigma_{xx}\frac{\partial^2}{\partial \bar{x}^2}a_{x,1} + 2\sigma_{xp}\frac{\partial^2}{\partial \bar{x}\,\partial \bar{p}}a_{x,1} + \sigma_{pp}\frac{\partial^2}{\partial \bar{p}^2}a_{x,1}\} + \ldots$$

$$\frac{d}{dt}\bar{p} = a_{p,1}(\bar{x},\bar{p}) + 0.5\{\sigma_{xx}\frac{\partial^2}{\partial \bar{x}^2}a_{p,1} + 2\sigma_{xp}\frac{\partial^2}{\partial \bar{x}\,\partial \bar{p}}a_{p,1} + \sigma_{pp}\frac{\partial^2}{\partial \bar{p}^2}a_{p,1}\} + \ldots$$

First-order approx.

Second-order correction

For the implementation we need a stochastic formalization of the price process:

• we assume an iid distributed liquidity component μ in excess demand with expectation 0

• prices change by one basic unit as a Poisson process with probabilities:

$$w_{\uparrow p} = \int_{-ED}^{\infty} \beta(ED + \mu)\, p(\mu)\, d\mu,$$

$$w_{\downarrow p} = \int_{-\infty}^{-ED} \beta(-ED - \mu)\, p(\mu)\, d\mu,$$

it follows that:

$$a_{p,1} = \sum_{x'} (p' - p) w_{p'p} = \beta ED$$

$$\Rightarrow \frac{d}{dt}\overline{p} = \overline{\beta ED} = \beta(\overline{x}T_c + (p_f - \overline{p})T_f)$$

Note:

- first order approx. of price equation is already exact because of linearity!

- the first order approximation to the bi-variate stochastic model coincides with our previous heuristically motivated *x*-*p*-dynamics

Second moments:

$$\overline{x_t^2} = \sum_x \sum_p x^2 Q(x,p;t), \overline{p_t^2} = \sum_x \sum_p p^2 Q(x,p;t),$$

$$\overline{xp} = \sum_x \sum_p xpQ(x,p;t)$$

$$\frac{d}{dt}\overline{x_t^2} = \sum_x \sum_p x^2 \frac{dQ(x,p;t)}{dt}$$

$$= \sum_x \sum_p x^2 \sum_{x'} (w_{xx'} Q(x',p;t) - w_{xx'} Q(x,p;t))$$

$$= \sum_x \sum_p \sum_{x'} ((x'-x)^2 + 2x(x'-x)) w_{x'x} Q(x,p;t) =$$

$$\overline{a_{x,2}(x,p)} + 2\overline{x a_{x,1}(x,p)}$$

Second moments…cont'd:

$$\frac{d}{dt}\overline{p_t^2} = \overline{a_{p,2}(x,p)} + 2\overline{pa_{p,1}(x,p)}$$

$$\frac{d}{dt}\overline{xp} = \sum_x \sum_p \; xp\Big(\sum_{x'}(w_{xx'}Q(x',p;t) - w_{x'x}Q(x,p;t)) +$$

$$\sum_{x'}(w_{pp'}Q(x,p';t) - w_{p'p}Q(x,p;t))\Big)$$

$$= \overline{a_{x,1}p} + \overline{a_{p,1}x}$$

Exact equations of variances and covariances:

$$\frac{d}{dt}\sigma_x^2 = \overline{a_{x,2}(x,p)} + 2\overline{(x-\overline{x})a_{x,1}}$$

$$\frac{d}{dt}\sigma_{xp} = \overline{(p-\overline{p})a_{x,1}} + \overline{(x-\overline{x})a_{p,1}}$$

$$\frac{d}{dt}\sigma_p^2 = \overline{a_{p,2}(x,p)} + 2\overline{(p-\overline{p})a_{p,1}}$$

First-order approximation to (co)variance dynamics:

$$\frac{d}{dt}\sigma_x^2 = a_{x,2}(\bar{x},\bar{p}) + 2\sigma_x^2 \frac{\partial}{\partial \bar{x}} a_{x,1} + 2\sigma_{xp} \frac{\partial}{\partial \bar{p}} a_{x,1}$$

$$\frac{d}{dt}\sigma_{xp} = \sigma_{xp}\left(\frac{\partial}{\partial \bar{x}} a_{x,1} + \frac{\partial}{\partial \bar{p}} a_{p,1}\right) + \sigma_p^2 \frac{\partial}{\partial \bar{p}} a_{x,1} + \sigma_x^2 \frac{\partial}{\partial \bar{x}} a_{p,1}$$

$$\frac{d}{dt}\sigma_p^2 = a_{p,2}(\bar{x},\bar{p}) + 2\sigma_p^2 \frac{\partial}{\partial \bar{p}} a_{p,1} + 2\sigma_{xp} \frac{\partial}{\partial \bar{x}} a_{p,1}$$

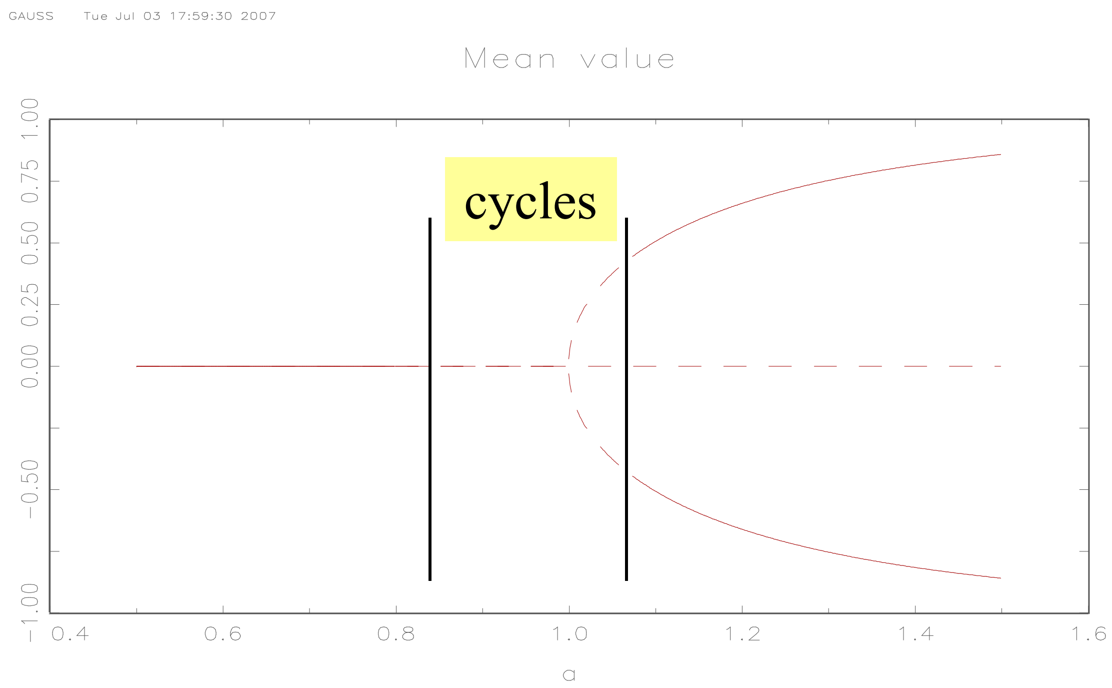
Closed system of 5 equations in 1st and 2nd moments!

Insights

- Variance equations from agent-based models generically contain autoregressive elements: nonlinear dynamics in 2nd moments, ARCH effects (Ramsey, 1996)

- Analysis of variances in stationary state, transient dynamics etc

• We can solve for stationary variances and covariances around the fundamental and bubble equilibria from our system



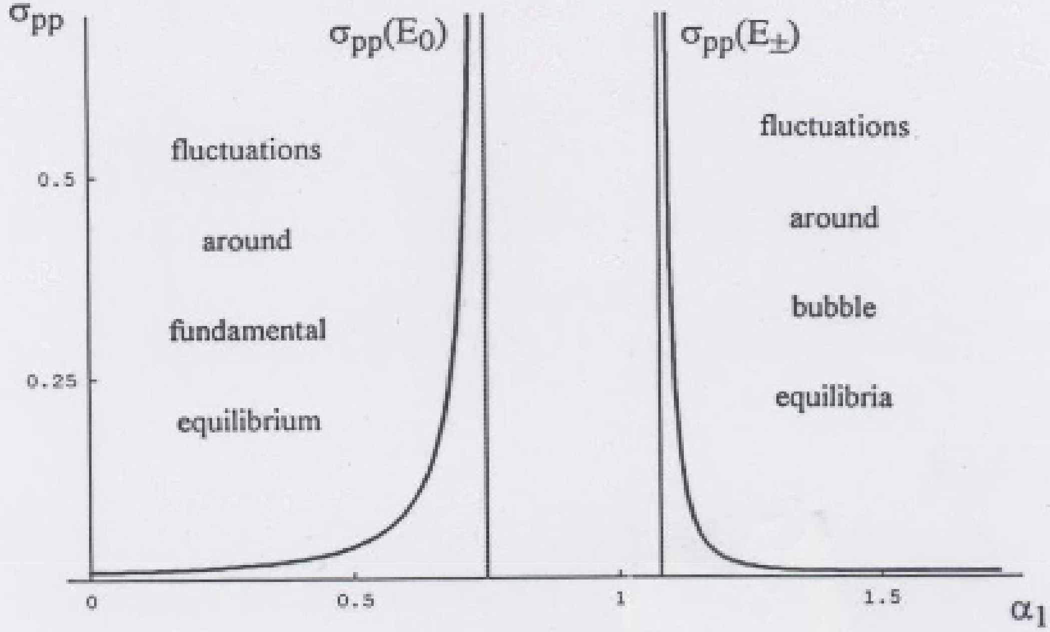


Fig. 1: Variance of prices in statistical equilibrium. The left-hand part shows the variance of prices in fundamental equilibrium (given in eq. 4.2c), the right-hand part shows the variance of prices in bubble equilibria (given in eq. 4.3c). The vertical asymptotes correspond to values of α_1 where either the fundamental equilibrium or the bubble equilibria loose stability. In the intermediate part between both asymptotes, no attracting state exists and only periodic solutions prevail. The parameter values used in this graph are: $v = 0.5$, $\alpha_2 = 0.75$, $\beta = 1$, $T_c = T_f = 0.5$, $N = 100$, $\sigma_\mu = 0.01$. Loss of stability occurs at $\alpha_1 = 0.75$ for the fundamental equilibrium and at $\alpha_1 \approx 1.08$ for bubble equilibria.

Same behavior for all 2nd moments!

Transient Dynamics

- Shock to fundamental value: overshooting and mean reversion
- Goes along with predictable variation in volatility: decrease of volatility upon impact, oscillatory reversion to steady state level

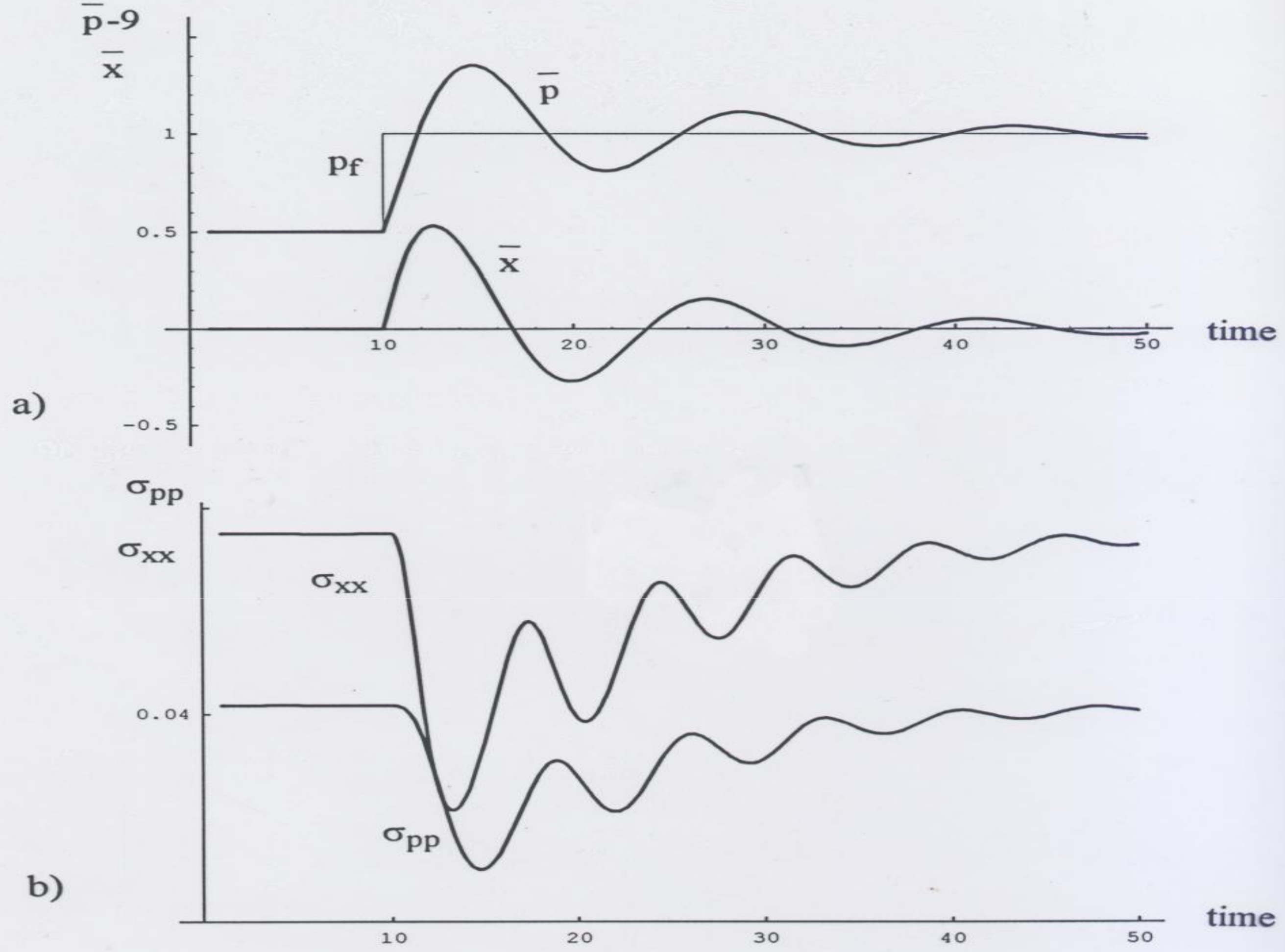


Fig. 2: Transient dynamics after shock to mean values. The shock consists in an increase of the fundamental value from 9.5 to 10 at time t = 10. Panel a) shows the dynamics of mean values $\overline{p}$ and $\overline{x}$. The thin line depicts the variation of p_f. Panel b) shows the transient dynamics of variances σ_{xx} and σ_{pp}. Most parameter values are the same as in Fig. 1 with the exceptions of $\sigma_\mu = 10^{-4}$ and $\alpha_1 = 0.6$. Equilibrium values of variances are $\sigma_{xx}(E_0) \approx 0.075$ and $\sigma_{pp}(E_0) \approx 0.042$.

Volatility shocks: persistence

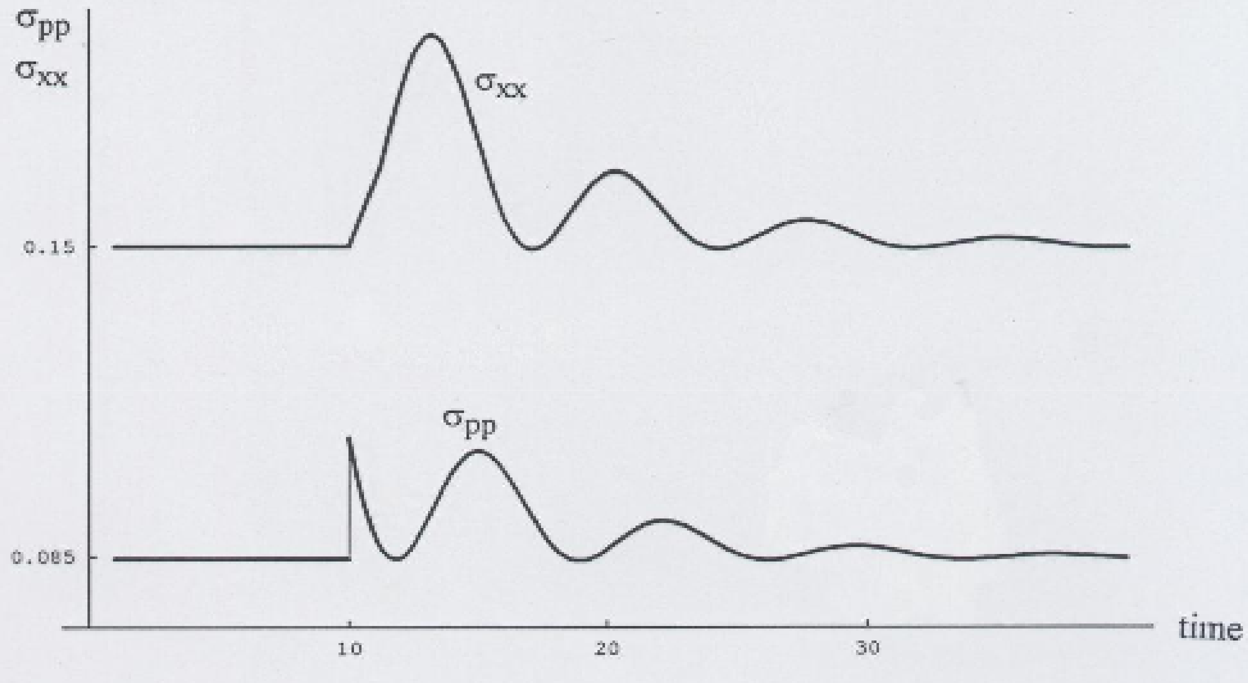


Fig. 3: Transient dynamics after volatility shock. The shock consists in a temporary increase of the variance σ_{pp} by about 25% at time t = 10. Note that mean values remain constant during this adjustment process. Parameter values are the same as in Fig. 1 together with $\alpha_1 = 0.6$. Equilibrium values of variances are $\sigma_{xx}(E_0) \approx 0.15$ and $\sigma_{pp}(E_0) \approx 0.085$.

Variances in Oscillatory Regime

- Cyclical variation of 2nd moments along the cycle
- Different degrees of predictability in different market phases
- Secular increase of theoretical variances because of random phase shifts

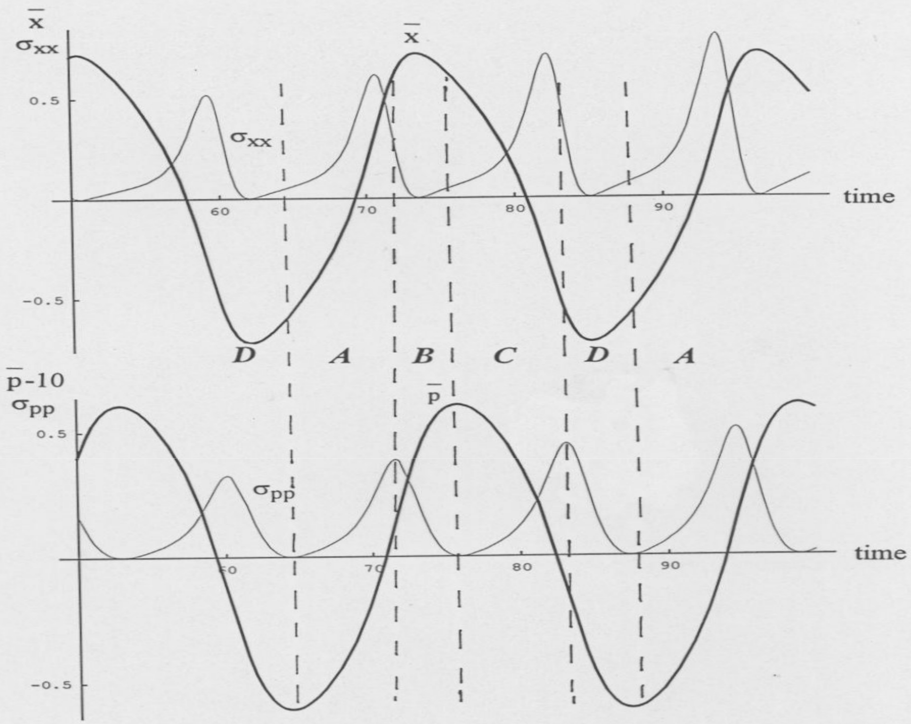


Fig. 4: Oscillatory dynamics. Part a) shows the dynamics of $\overline{x}$ and σ_{xx}, part b) shows $\overline{p}$ and σ_{pp}. Both variances had to be scaled appropriately in order to lodge them in the same graph as mean values. Parameter values are as in Fig. 1 together with $\alpha_1 = 1$.

Variances in Oscillatory Regime

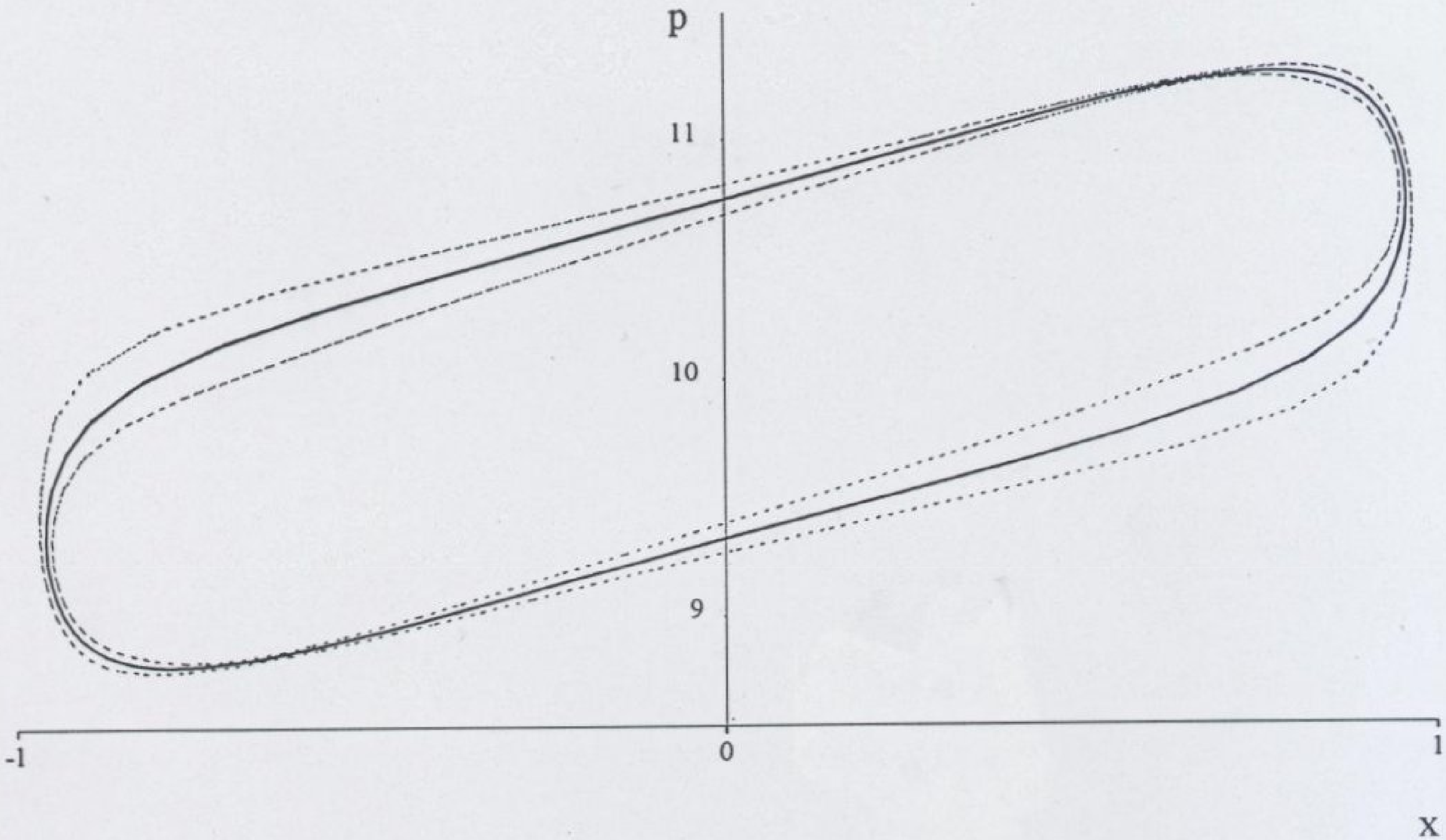


Fig. 5: Limit cycle and limit gully. The thick, solid line shows the periodic solution obtained for mean values $\overline{x}$ and $\overline{p}$. The broken lines represent the varying extent of "transversal" fluctuations away from the orbit during the different stages of the cycle (the "limit gully").

A Richer Framework (Lux and Marchesi, 1999, 2000)

As before:

- different types of traders: "***noise traders***" and "***fundamentalists***"
- noise traders rely on: ***charts*** (price trend) and ***flows*** (behavior of others)
- traders formulate demand and supply, market maker adjusts the price in the usual manner

New features:

- ✓ traders compare profits gained by noise traders and fundamentalists and ***switch to the more successful group***.
- ✓ changes of the (log of the) fundamental value follow a Wiener process: $\ln(p_{f,t}) = \ln(p_{f,t-1}) + \varepsilon_t \Delta t$ with $\varepsilon_t \sim N(0, \sigma_\varepsilon)$

  -> ***news arrival process exhibits neither fat tails nor clustered volatility*** still: changes of behavior occur according to ***state-dependent transition probabilities***:

this means: during a small time increment Δt, one individual will switch between behavioral alternatives (i and j, say) with probability: $\pi_{ij}(t)\, \Delta t$

asynchronous reactions of

individual agents:

(1) switches of noise traders between optimistic and pessimistic

(2) adjustment of the price [by one elementary unit, e.g. one cent] depending on imbalances between demand and supply.

transition probabilities:

$p_{+-} = v_1 \, exp(U_1)$ and $p_{-+} = v_1 \, exp(-U_1)$,

$$w_{\uparrow p} = \int_{-ED}^{\infty} \beta(ED + \mu) p(\mu) d\mu, \quad w_{\uparrow p} = \int_{-\infty}^{-ED} \beta(-ED - \mu) p(\mu) d\mu,$$

New component:

switches between noise traders and fundamentalists depending on

comparison of profits:

actual profits gained by chartists: capital gains (or losses) *vs.*

expected profits of fundamentalists: percentage difference between

prevailing price and assumed fundamental value

transition rates:

$$\pi_{nf} = v_2 \, exp(U_2) \quad \text{and} \quad \pi_{fn} = v_2 \, exp(-U_2),$$

$$\text{with: } U_2 = \alpha_3 * \textit{profit differential}$$

Profit differentials in transition rates between noise trading and fundamentalism:

$$U_{2,1} = \alpha_3 \{ \underbrace{\frac{r + \frac{1}{v_2} p'(t)}{p}}_{\substack{\textit{profit of chartists} \\ \textit{from } n_+ \textit{ group}}} - R - \underbrace{s \cdot \left| \frac{p_f - p}{p} \right|}_{\textit{fundamentalists' profit}} \}$$

$$U_{2,1} = \alpha_3 \{ R - \underbrace{\frac{r + \frac{1}{v_2} p'(t)}{p}}_{\substack{\textit{profit of chartists} \\ \textit{from } n_- \textit{ group}}} - \underbrace{s \cdot \left| \frac{p_f - p}{p} \right|}_{\textit{fundamentalists' profit}} \}$$

R: riskfree interest rate, r: nom dividend so that $r/p_f = R$ (risk neutrality)

Mean value dynamics

$$\frac{d}{dt}\bar{x} = \ldots, \; for \; x = \frac{n_+ - n_-}{n_c}$$

Sentiment index

$$\frac{d}{dt}\bar{z} = \ldots, \; for \; z = \frac{n_c}{N}$$

Fraction of noise traders

$$\frac{d}{dt}\bar{p} = \beta(ED_f + ED_c) = \beta(xzT_c + T_f (1 - z)(p_f - p))$$

Results: Existence of Equilibria

Proposition 1:

(a) The mean-value dynamics of x, p and z possesses the following stationary solutions:

(i) $x^* = 0$, $p^* = p_f$ with arbitrary z,

(ii) $x^* = 0$, $z^* = 1$ with arbitrary p,

(iii) $z^* = 0$, $p^* = p_f$ with arbitrary x;

(b) no stationary states with both $x^* \neq 0$ and $p^* \neq pf$ exist.

Proof: by mean-field approximation

Absorbing states: relatively uninteresting

Note: no more bubble equilibria!!

Interpretation of Theoretical Results

Results for the dynamics of mean-values for the price and the number of individuals in each subgroup:

a continuum of a stationary states exists which are characterized by:

(i) price = fundamental value (on average),

(ii) balanced disposition among noise traders

(iii) as in equilibrium noise traders and fundamentalists perform equally well: composition of the population is ***indeterminate***

Results: Stability of Equilibria

Proposition 2:

An equilibrium on the line ($x^* = 0$, $p^* = p_f$, z^* arbitrary) is unstable (repelling) if one of the following conditions is violated:

(cond 1)

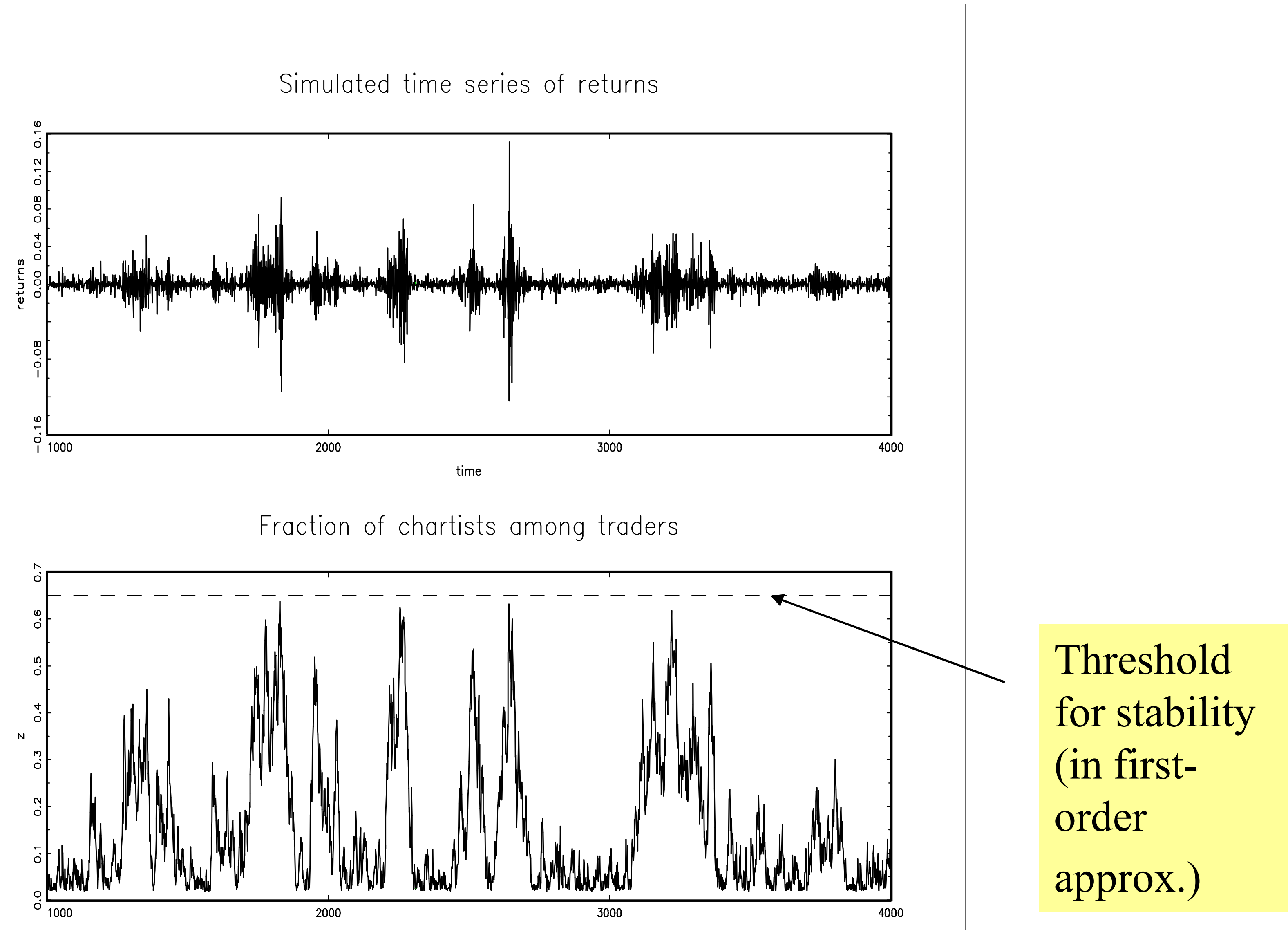
$$2z^* v_1(\alpha_1 + \alpha_2 \frac{\beta}{v_1} z^* T_c - 1) + $$
$$+2(1 - z^*)\alpha_3 \beta z^* T_c / p_f - \beta(1 - z^*)T_f < 0$$

(cond 2)

$$\alpha_1 < 1 - \alpha_3 \frac{v_2 T_c R}{v_1 T_f p_f}$$

-> from these conditions one can compute an interval of stable equilibria: $z^* \in (0, \bar{z}]$

Proof: by mean-field approximation



Example of the Dynamics: returns and simultaneous development of the fraction of chartists, z. The broken line indicates the critical value at which a loss of stability occurs.

Interpretation of Theoretical Results… cont'd

Though the system always tends towards a stable equilibrium, it experiences sudden transient ***phases of destabilization***.

What happens can be understood as a local bifurcation:

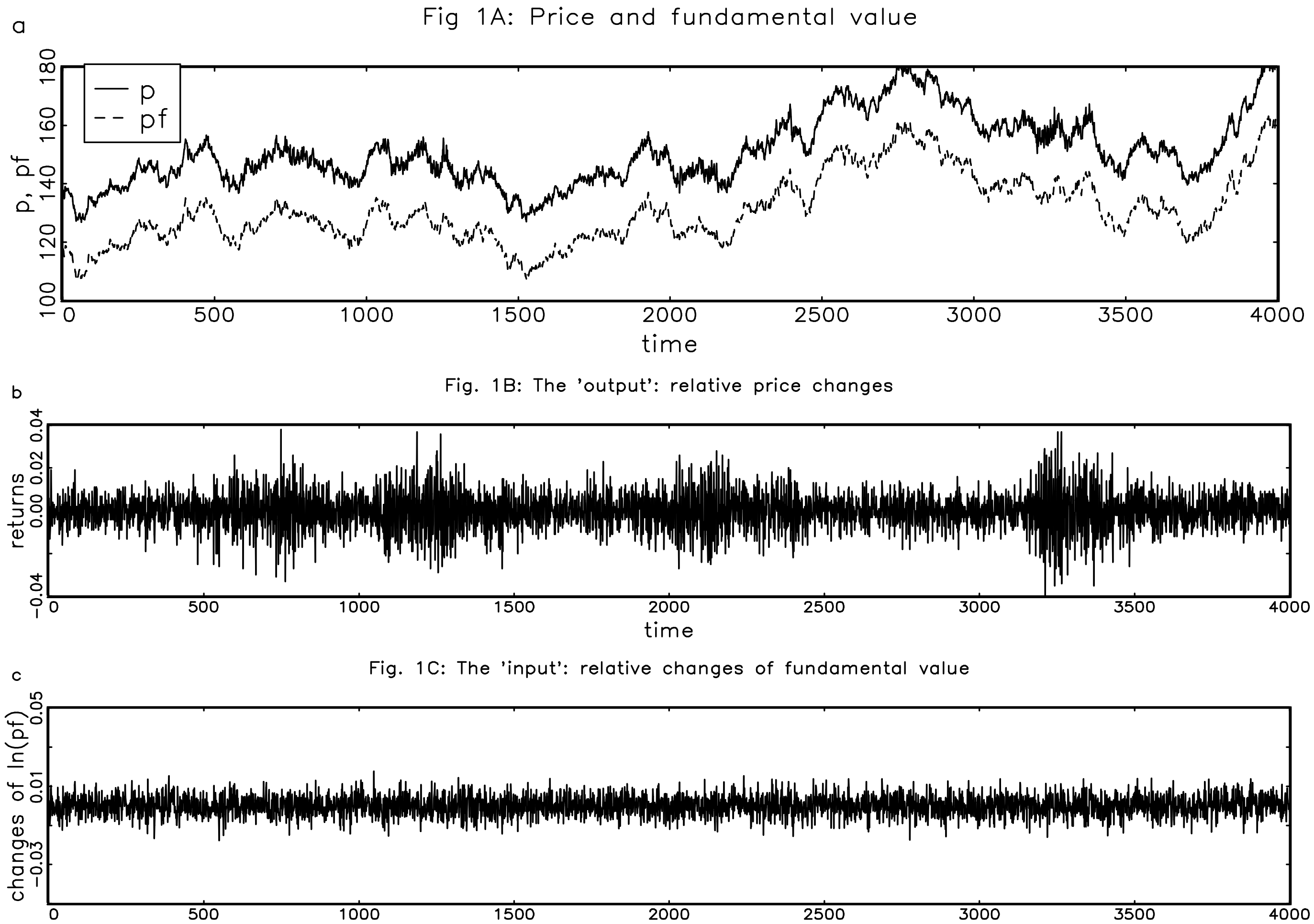
due to the stochastic nature of the model there is always some noise with ***most of the time***: only minor fluctuations around the equilibrium,

↓

however: ***stability*** of the equilibrium depends on the fraction of noise traders present,

↓

every once in a while, stochastic motion or extraneous forces (*news*!) will push the system beyond the stability threshold: onset of severe, but short-lived fluctuations.



Typical snapshot from a simulation run. The upper panel depicts the market price p (solid line) and the fundamental value p_f (dotted line). The latter series has been shifted vertically for better visibility. The middle and bottom panel show returns and log changes of the fundamental value, respectively.

Stylized Facts as Emergent Phenomena in Multi-Agent Systems

Efficient Markets vs. Interacting Agents

EMH: prices *immediately* reflect forthcoming news

-> statistical characteristics of financial returns are *a mere reflection* of similar characteristics of the *news arrival process*

Interacting Agent Hypothesis: dynamics of asset returns arise endogenously from the trading process,

market interactions ***magnify and transform*** exogenous news into fat tailed returns with clustered volatility

A Stochastic Framework for Socio-Economic Interactions

The case of two interacting populations

Assume there are two groups with members *2N* and *2M* respectively and two opinions (strategies) "1" and "2" within each group:

$$2M = m_1 + m_2\ ,\ 2N = n_1 + n_2$$

The configuration of the overall population consists of the group occupation numbers $\{m_1, m_2, n_1, n_2\}$ or more compactly $\{m, n\}$ with $m = \frac{m_1 - m_2}{2}, n = \frac{n_1 - n_2}{2}$.

Movements between subgroups could depend on the distribution of "1" and "2" attitudes within the same population, but might also be influenced by the distribution of attitudes within the second group.

Individual transition rates might then be written as:

$$p^{\mu}_{12} = V_{\mu}\, exp\left(\delta^{\mu} + \kappa^{\mu\mu} m + \kappa^{\mu\nu} n\right) = V_{\mu}\, exp\left(\Delta u^{\mu}(m,n)\right)$$

$$p^{\mu}_{21} = V_{\mu}\, exp\left(-\delta^{\mu} - \kappa^{\mu\mu} m - \kappa^{\mu\nu} n\right) = V_{\mu}\, exp\left(-\Delta u^{\mu}(m,n)\right)$$

$$p^{\nu}_{12} = V_{\nu}\, exp\left(\delta^{\nu} + \kappa^{\nu\mu} m + \kappa^{\nu\nu} n\right) = V_{\nu}\, exp\left(\Delta u^{\nu}(m,n)\right)$$

$$p^{\nu}_{21} = V_{\nu}\, exp\left(-\delta^{\nu} - \kappa^{\nu\mu} m - \kappa^{\nu\nu} n\right) = V_{\nu}\, exp\left(-\Delta u^{\nu}(m,n)\right)$$

Two interacting populations

One can again set up Master and Fokker-Planck equations for the bi-variate {m, n} dynamics, and derive the exact law of motion for the mean values. Their first-order approximation leads to:

$$\frac{d\bar{m}}{dt} = 2V_\mu \left\{ M\,Sinh\left(\Delta u^\mu\left(\bar{m},\bar{n}\right)\right) - \bar{m}\,Cosh\left(\Delta u^\mu\left(\bar{m},\bar{n}\right)\right) \right\}$$

$$\frac{d\bar{n}}{dt} = 2V_\nu \left\{ N\,Sinh\left(\Delta u^\nu\left(\bar{m},\bar{n}\right)\right) - \bar{n}\,Cosh\left(\Delta u^\nu\left(\bar{m},\bar{n}\right)\right) \right\}$$

Equivalently: dynamics for opinion indices y = m/M, x = n/N

The possibility of spillovers between groups allows for a rich variety of outcomes. Consider the simple version:

$$\delta^{\mu} = \delta^{\nu} = 0, \kappa^{\mu\mu} = \kappa^{\nu\nu} = \tilde{\kappa}, M = N, V_{\mu} = V_{\nu},$$

and define $\tilde{\kappa}^{\mu\nu} = \kappa^{\mu\nu} M, \tilde{\kappa}^{\nu\mu} = \kappa^{\nu\mu} N.$

The dynamics of the opinion indices becomes:

$$\frac{d\overline{m}}{dt} = sinh\left(\tilde{\kappa}\overline{m} + \tilde{\kappa}^{\mu\nu}\overline{n}\right) - \overline{m}\,cosh\left(\tilde{\kappa}\overline{m} + \tilde{\kappa}^{\mu\nu}\overline{n}\right)$$

$$\frac{d\overline{m}}{dt} = sinh\left(\tilde{\kappa}^{\nu\mu}\overline{m} + \tilde{\kappa}\overline{n}\right) - \overline{n}\,cosh\left(\tilde{\kappa}^{\nu\mu}\overline{m} + \tilde{\kappa}\overline{n}\right)$$

Weak within-group herding and weak positive between-group interaction

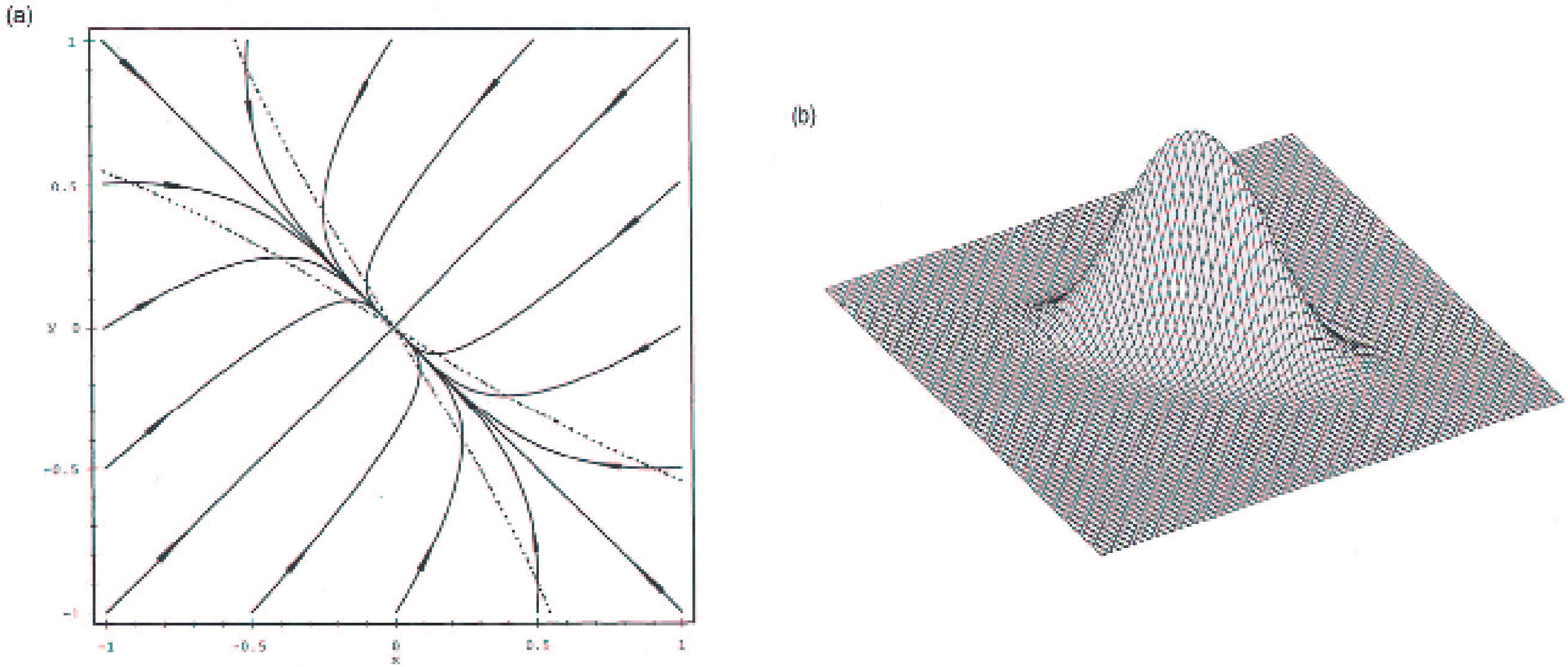


Figure 1: (a) Parameters $\widetilde{\kappa} = 0.2$ and $\widetilde{\kappa^{\nu\mu}} = 0.5$. Weak internal agglomeration trend and weak symmetrical reciprocal segregation trend. All fluxlines approach thr origin $(0, 0)$ which describes the homogenous mixture of populations $\mathcal{P}^{\mu}$ and $\mathcal{P}^{\nu}$ and is the only stable stationary point; (b) Parameters as in Figure (a). $2N = 80$; Unimodal stationary propability distribution peaked around the stable origin $(0, 0)$.
Source: Weidlich (2000), p. 90.

Weak within-group herding and strong negative between-group interaction

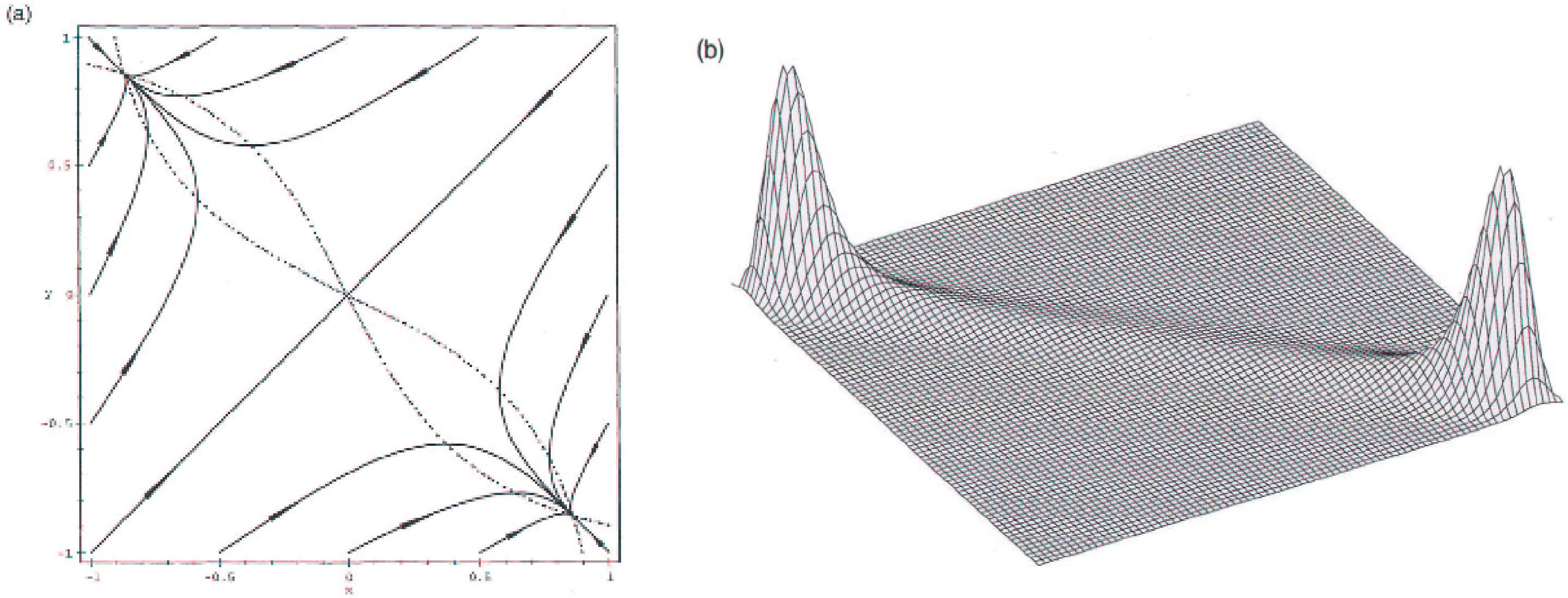


Figure 2: (a) Moderate internal agglomeration trend $\widetilde{\kappa} = 0.5$ and strong reciprocal segregation trend $\widetilde{\sigma} = 1.0$. The two stable stationary points in the second and fourth quadrant describe stable segregation of populations $\mathcal{P}^{\mu}$ and $\mathcal{P}^{\nu}$ in separate "ghettos". The fluxlines approach one of these stable equlibrium points; (b) Parameters as in Figure (a), $2N = 80$. The bimodal stationary probability distribution is peaked around the stationary points.
Source: Weidlich (2000), p. 92.

$$\widetilde{\sigma} = -\widetilde{\kappa}^{\mu\nu} = -\widetilde{\kappa}^{\nu\mu}$$

Weak within-group herding and strong asymmetric between-group interaction

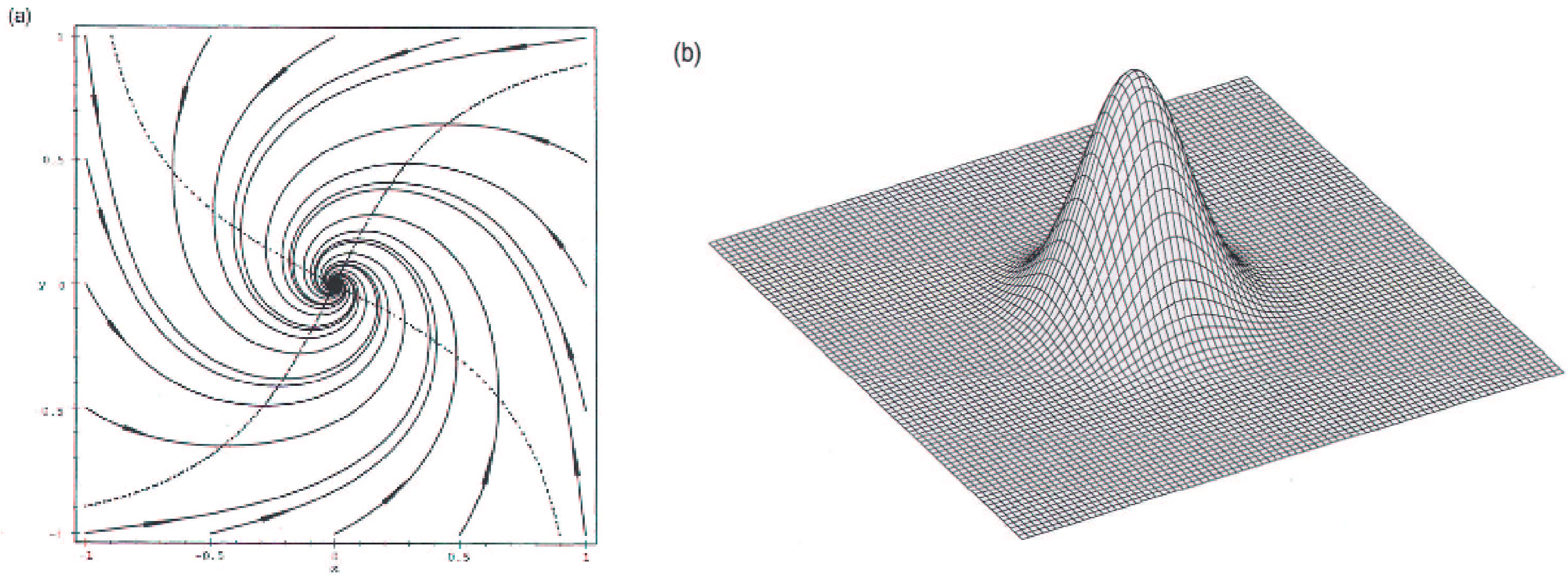


Figure 3: (a) Moderate internal agglomeration trend $\widetilde{\kappa}$ and strong asymmetric interaction $\widetilde{\kappa^{\mu\nu}} = -1.0$ and $\widetilde{\kappa^{\nu\mu}} = +1.0$. There exists one stable focus, the orgin (0,0) into which all fluxlines spiral; (b) Parameters as in (a), $2N = 80$. The unimodal stationary probability distributions is peaked around the stable focus (0,0).

Source: Weidlich (2000), p. 93.

Strong within-group herding and strong negative between-group interaction

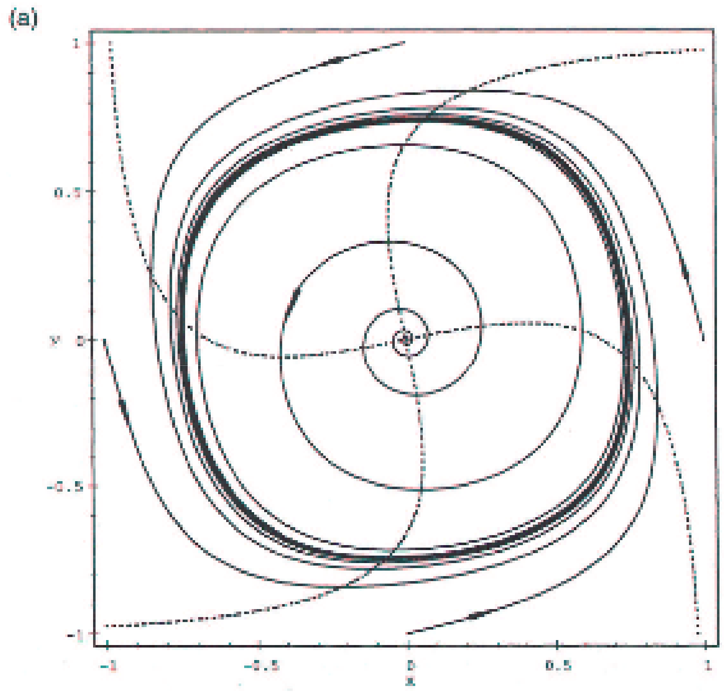


Figure 4: (a) Very strong internal agglomeration trend $\widetilde{\kappa} = 1.2$ and strong asymmetric interaction $\widetilde{\kappa^{\mu\nu}} = -1.0$ and $\widetilde{\kappa^{\nu\mu}} = +10$. The orgin (0,0) is an unstable focus. All fluxlines approach a limit cycle.
Source: Weidlich (2000), p. 94.

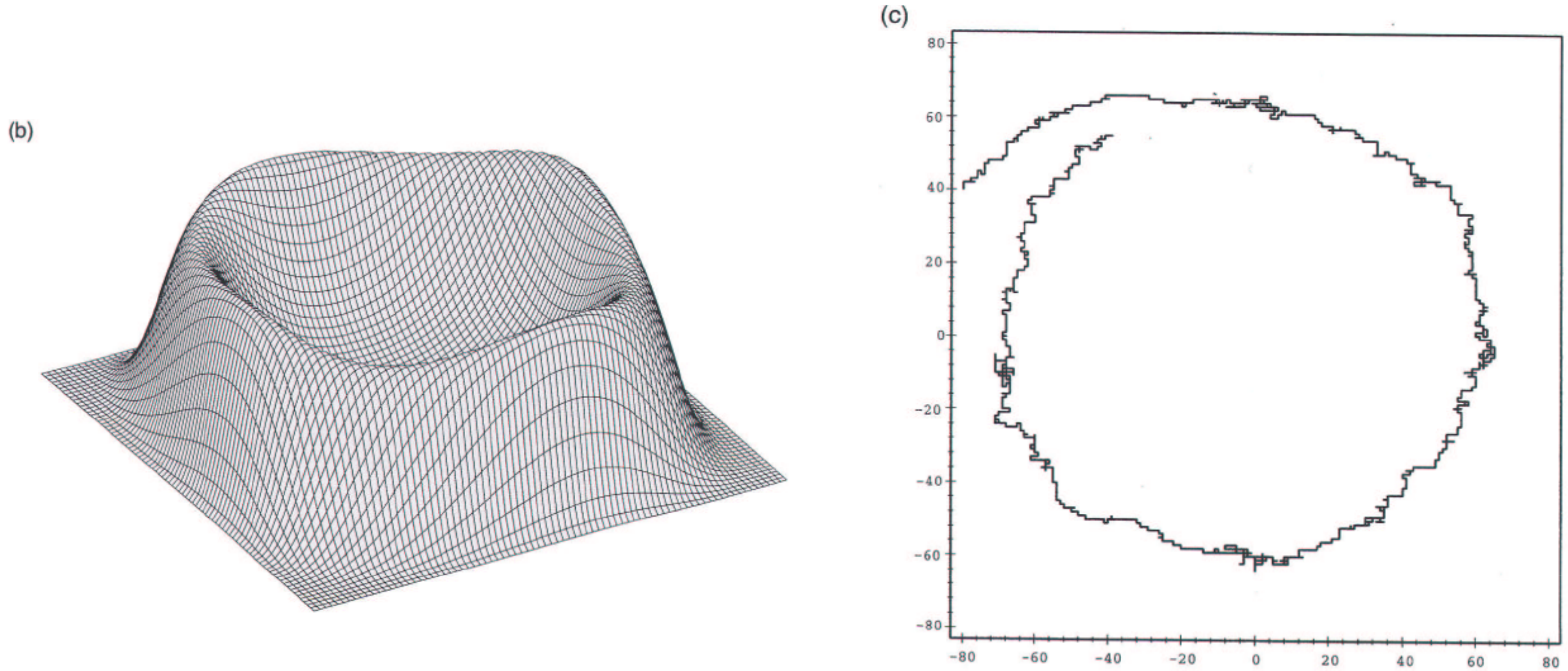


Figure 5: (b) Parameters in Fig.4(a), $2N = 80$. The quadrumodal stationary probability has four maxima corresponding to metastable situations and ridges between the maxima along the limit cycle; (c) Parameters as in Fig.4(a) and (b). example of stochastic trajectory belonging to transition rates. The trajectory abides around the metastable points of maximal probability and traverses at fast pace the states between the metastable situations.
Source: Weidlich (2000), p. 95.

Estimation: for a time series of discrete observations X_s of our canonical process, the likelihood fun

$$-log f_0(X_0 \mid \theta) - \sum_{s=0}^{T-1} log f(X_{s+1} \mid X_s, \theta)$$

with discrete observations X_s, the Master or FP equations are the exact or approximate laws of motion for the transient density and allow to evaluate log $f(X_{s+1}|X_s,\theta)$ and , therefore, to estimate the parameter vector θ ($\theta = (v, \alpha_0,\alpha_1)'$)!

Implementation

- Usually no analytical solution for transient pdfs from Master or FP equations
- Numerical solution of Master equation too computation intensive if there are many states x (i.e., particularly with large N)
- Numerical solutions of FP equation is less computation intensive, various methods available for discretization of stochastic differential equations

Finite Difference Approximation

$$\frac{\partial p_x}{\partial t} = \frac{\partial}{\partial x}\mu(x)p_x + \frac{\partial^2}{\partial x^2}g(x)p_x$$

→

Space-time grid:
$x_{min} + jh,\ t_0 + ik$

$$\frac{p_j^{i+1} - p_j^i}{k} = \frac{\mu_{j+1}p_{j+1}^i - \mu_j p_j^i}{h} + \frac{g_{j+1}p_{j+1}^i - 2g_j p_j^i + g_{j-1}p_{j-1}^i}{h^2}$$

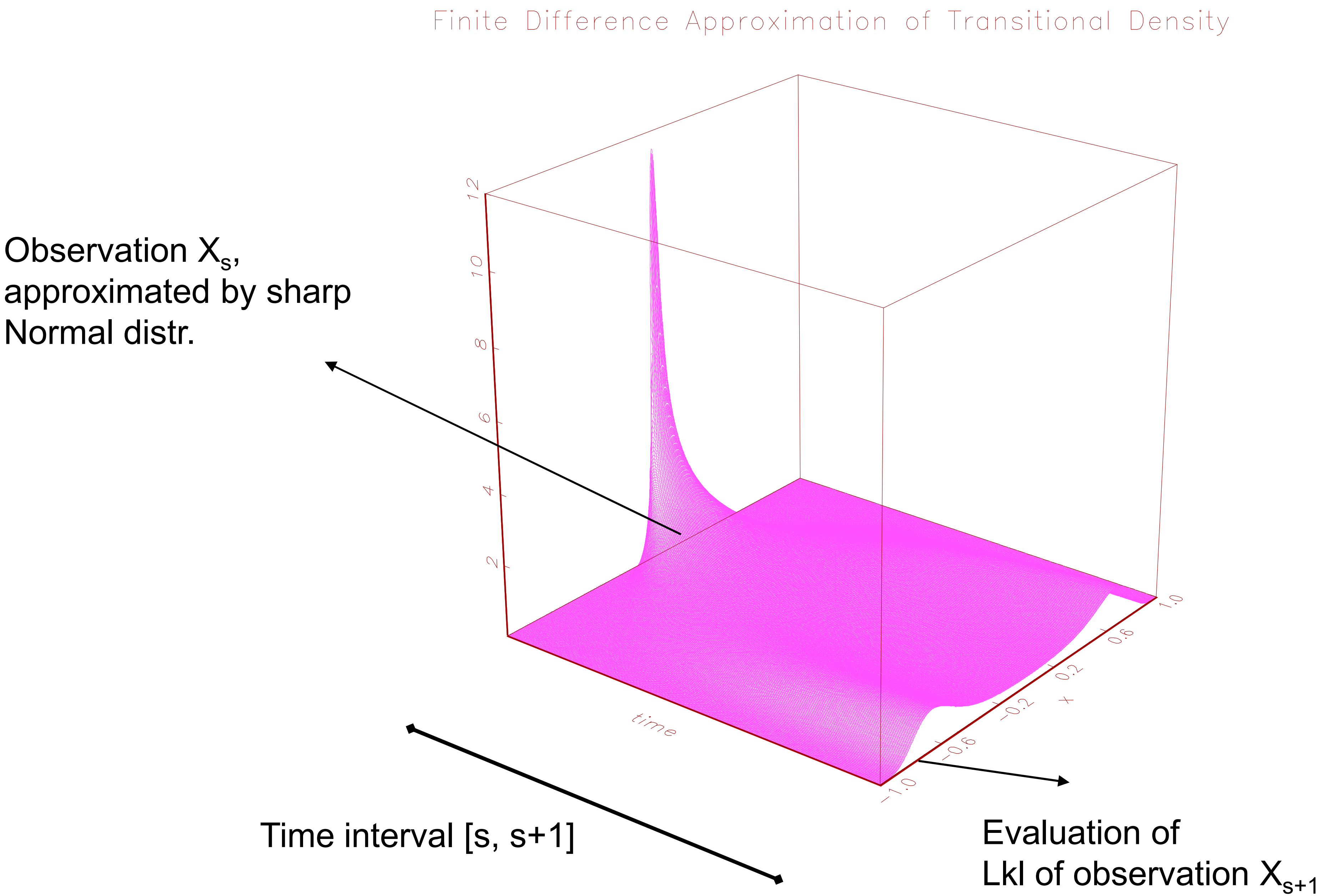
forward difference

$$\frac{p_j^i - p_j^{i-1}}{k} = \frac{\mu_{j+1}p_{j+1}^i - \mu_j p_j^i}{h} + \frac{g_{j+1}p_{j+1}^i - 2g_j p_j^i + g_{j-1}p_{j-1}^i}{h^2}$$

backward difference

Numerical Solution of FPE

- Forward and backward approximations are of first-order accuracy: combining them yields *Crank-Nicolson* scheme with second-order accuracy -> solution at intermediate points $(i+1/2)k$ and $(j+1/2)h$
- This allows to control the *accuracy of ML estimation*: estimates are consistent, asymptotically normal and asymptotically equivalent to complete ML estimates (Poulsen, 1999)


Finite Difference Approximation of Transitional Density
Observation X_s, approximated by sharp Normal distr.
12
10
8
6
4
2
time
x
−1.0
−0.6
−0.2
0.2
0.6
1.0
Time interval [s, s+1]
Evaluation of Lkl of observation X_{s+1}

Monte Carlo Experiments

- Does the method work in our case of a potentially bi-modal distribution, is it efficient for small samples?

- Do we have to go at such pains for the ML estimation? Couldn‘t we do it with a simpler approach (Euler approximation)?

Applicability to our famework: check the order of accuracy

Denote by v_1, v_2, v_3 the approximation errors from expansions using step sizes k and h, h/2, h/4

$$v_1 = f - hc - kd - h^2 l - k^2 m + \dots \tag{4}$$

$$v_2 = f - 0.5hc - kd - 0.25h^2 l - k^2 m + \dots \tag{5}$$

$$v_3 = f - 0.25hc - kd - 0.0625h^2 l - k^2 m + \dots \tag{6}$$

$$\frac{v_2 - v_1}{v_3 - v_2} = 2\frac{c + 1.5hl}{c + 0.75hl} \tag{7}$$

First order: the ratio is ~2,
Second order: it yields ~4

$x \| t$	0.25	0.5	0.75	1	1.25	1.5	1.75	2
-0.75	4.0	4.0	3.9	4.0	4.0	4.0	4.0	3.9
-0.5	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0
-0.25	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0
0	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0
0.25	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0
0.5	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0
0.75	4.0	4.0	3.9	4.0	4.0	4.0	4.0	3.9

h-ratio

$x \| t$	0.25	0.5	0.75	1	1.25	1.5	1.75	2
-0.75	4.0	3.9	4.0	4.0	4.0	4.0	4.0	4.0
-0.5	3.9	4.0	4.0	4.0	4.0	4.0	4.0	4.0
-0.25	3.9	4.0	4.0	4.0	4.0	4.0	4.0	4.0
0	6.1	4.0	4.0	4.0	4.0	4.0	4.0	4.0
0.25	3.9	4.0	4.0	4.0	4.0	4.0	4.0	4.0
0.5	3.9	4.0	4.0	4.0	4.0	4.0	4.0	4.0
0.75	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0

k-ratio

Table 1: Order determination for the Crank-Nicolson method applied to the interacting agent model. All parameter values and settings like in Fig. 1

Procedure works as expected, even for bi-modality

Monte Carlo Study of MLE with Crank-Nicolson Approximation

		Euler			Crank-Nicolson $k=1/8$			Crank-Nicolson $k=1/16$		
		v	α_0	α_1	v	α_0	α_1	v	α_0	α_1
$v=3$, $\alpha_0=0$, $\alpha_1=0.8$	mean	0.999	-0.001	0.642	2.980	-0.000	0.793	3.023	-0.000	0.794
	SSE	0.091	0.007	0.052	0.567	0.005	0.028	0.585	0.005	0.028
	RMSE	2.003	0.007	0.166	0.564	0.005	0.028	0.583	0.005	0.028
$\alpha_0=0.2$, $\alpha_1=0.8$	mean	0.439	0.578	0.123	2.992	0.216	0.772	3.547	0.211	0.782
	SSE	0.048	0.097	0.171	1.046	0.057	0.105	1.422	0.038	0.069
	RMSE	2.561	0.390	0.698	1.041	0.059	0.108	1.517	0.039	0.071
$\alpha_0=0$, $\alpha_1=1.2$	mean	1.019	0.000	1.173	2.884	0.000	1.196	2.952	0.000	1.196
	SSE	0.126	0.024	0.034	0.457	0.009	0.015	0.499	0.009	0.015
	RMSE	1.913	0.024	0.043	0.469	0.009	0.016	0.499	0.009	0.015
$\alpha_0=0.2$, $\alpha_1=1.2$	mean	0.232	1.741	0.698	1.369	0.262	1.123	1.748	0.245	1.144
	SSE	0.026	0.350	0.426	0.245	0.127	0.159	0.439	0.128	0.159
	RMSE	2.768	1.580	1.945	1.326	0.141	0.175	1.326	0.135	0.168

Table 3: Approximate ML Estimates

Results from Experiments

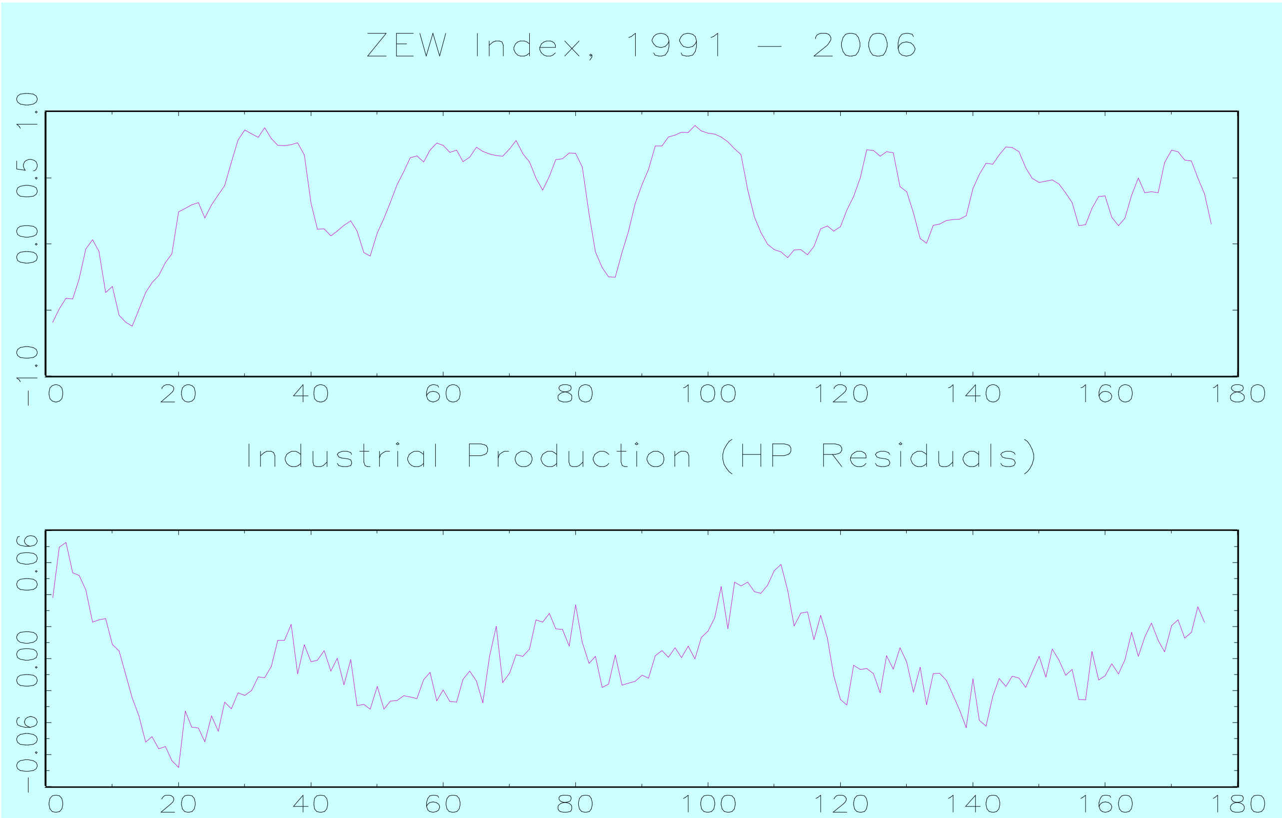
- Crank-Nicolson has expected order of accurary, works well in ML estimation
- Modest number of time steps (k) sufficient for high accuracy
- Implicit FDs have practically the same peformance as CN
- ML estimation also works well with endogenous N

Empirical Application

- The framework of the canonical model is close to what is reported in various *business climate indices*
  - *Germany: ZEW Indicator of Economic Sentiment,*
    Ifo Business Climate Index
  - *US: Michigan Consumer Sentiment Index,*
    Conference Board Index
  - ….

ZEW Index of Economic Sentiment, 1991 – 2006,

Monthly data, index = #positive - # negative, ca. 350 respondents



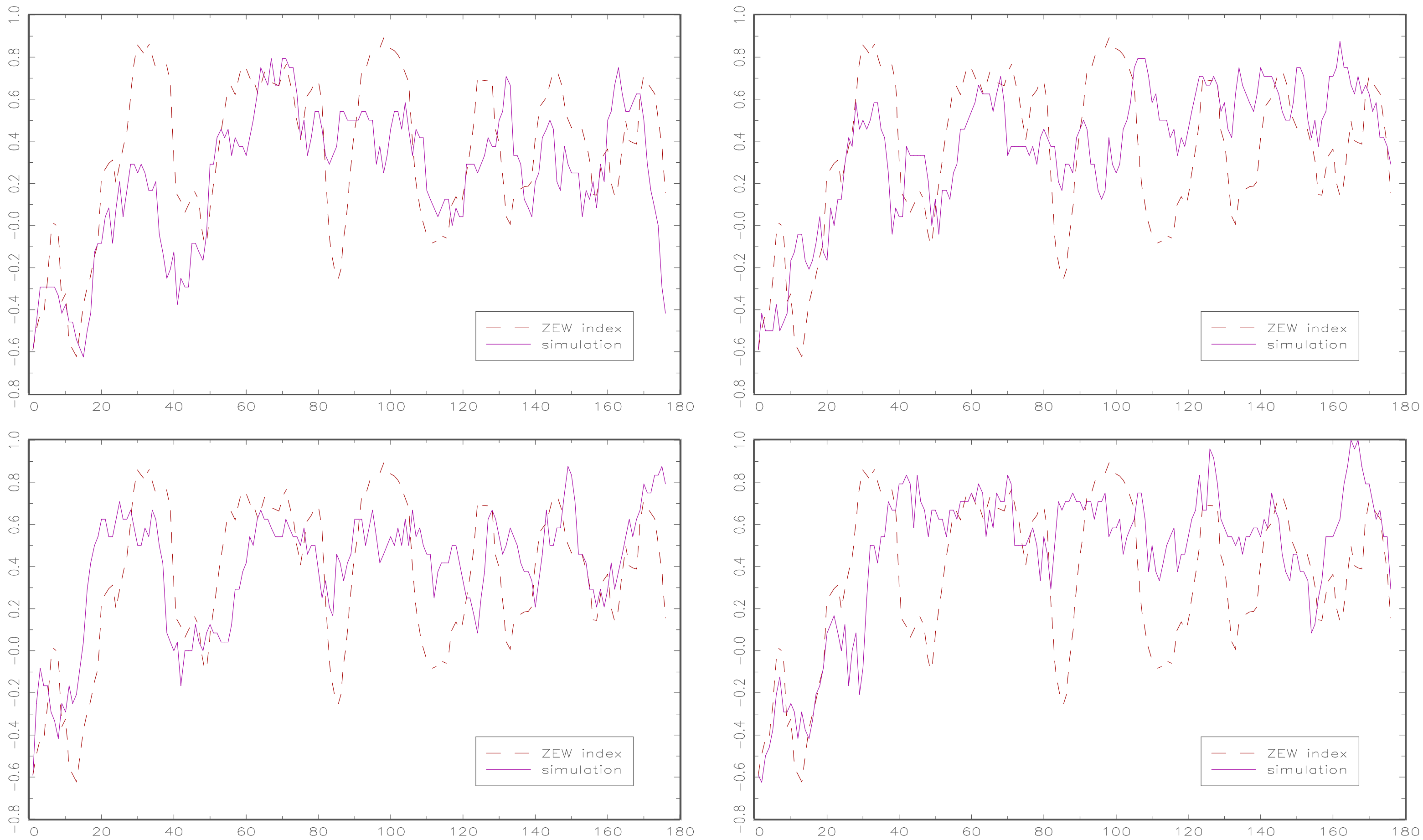
	v	α_0	α_1	α_2	α_3	N	ML	AIC
Model 1 (baseline)	0.78 (0.06)	0.01 (0.01)	1.19 (0.01)			*official: 350/2*	-726.9	1459.8
Model 2 (end. N)	0.15 (0.07)	0.09 (0.06)	0.99 (0.14)			21.21 (9.87)	-655.9	1319.7
Model 3 (feedback from IP)	0.13 (0.06)	0.09 (0.07)	0.93 (0.16)	-4.55 (2.53)		19.23 (8.78)	-650.4	1310.9
Model 4 (moment.)	0.14 (0.05)	0.10 (0.06)	0.91 (0.14)		2.11 (0.76)	27.24 (9.63)	-627.5	1265.1
Model 5 (mom. + IP)	0. 12 (0.05)	0.11 (0.06)	0.86 (0.16)	-2.82 (1.65)	2.23 (0.81)	25.12 (8.95)	-624.9	1261.94

Extensions of Baseline Model

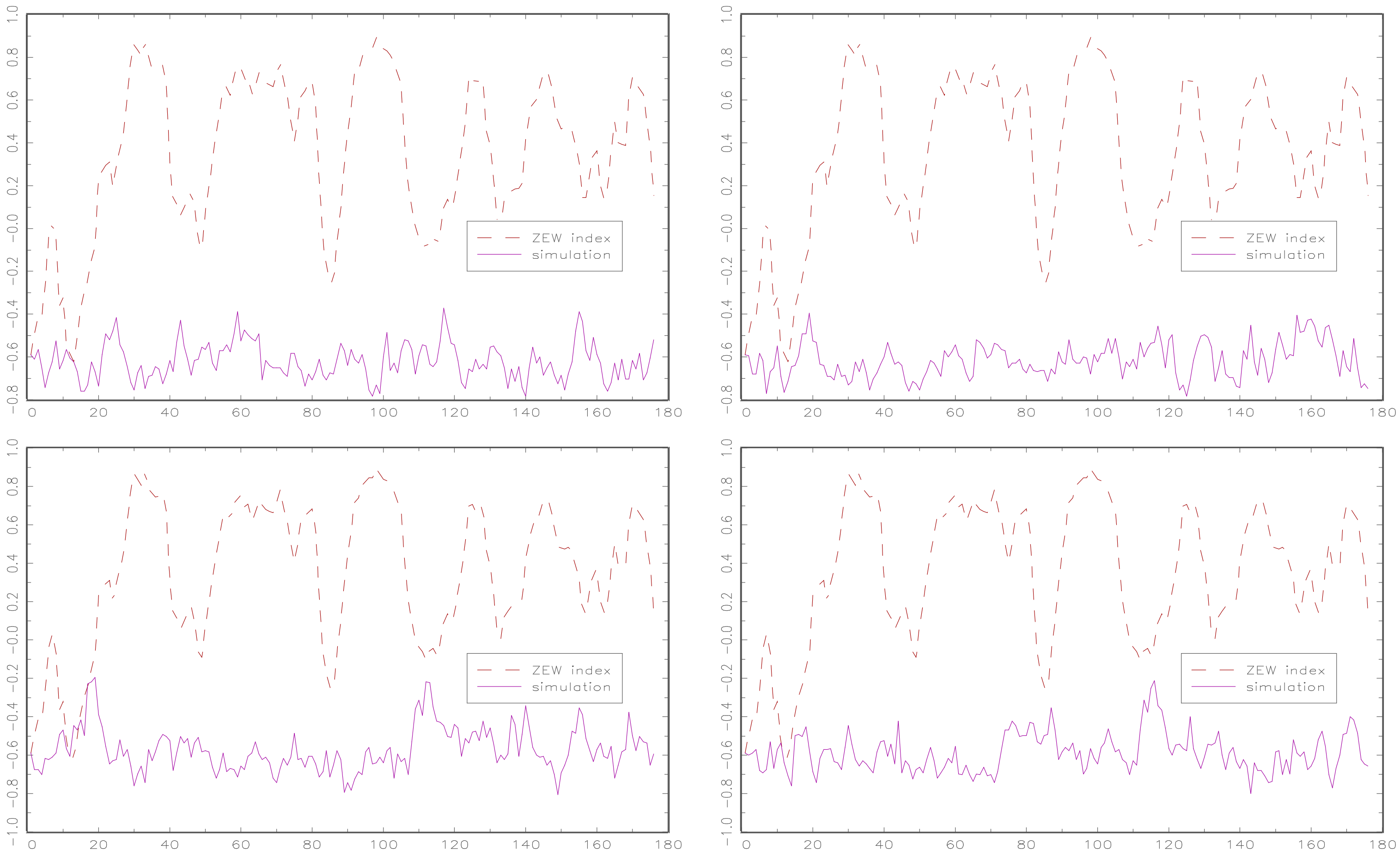
- introduction of exogenous variables (industrial production, interest rates, unemployment, political variables,…)
- ‘momentum’ effect
- endogenous N: ‘effective’ number of independent agents

$$U_t = \alpha_0 + \alpha_1 x_t + \alpha_2 IP + \alpha_3 (x_t - x_{t-1})$$

... a few simulations of model V (identical starting value of x, identical influence from IP

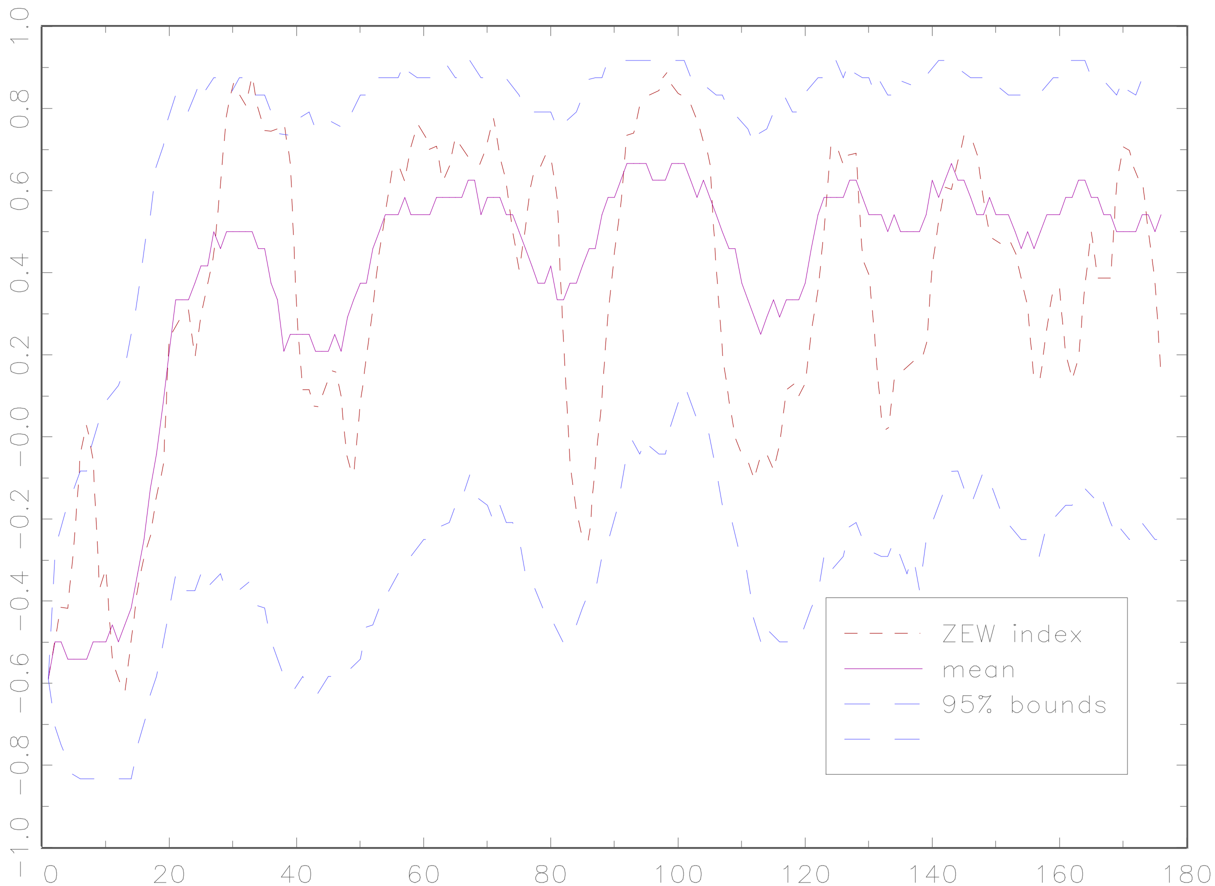


For comparison: simulations of model I (identical starting value of x) -> no similarity

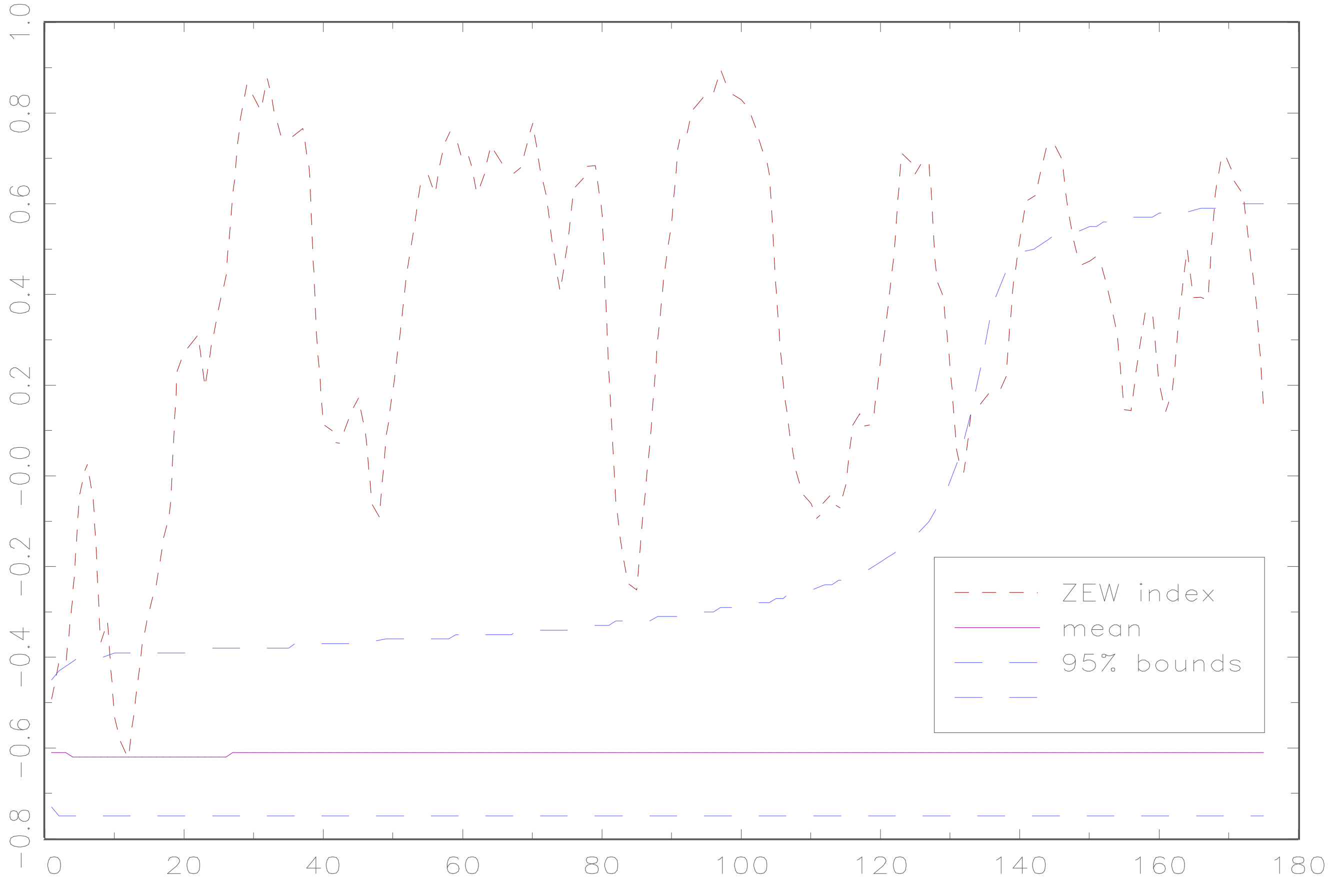


Specification tests: Mean and 95% confidence interval from model 3

(conditional on initial condition and influence form IP)

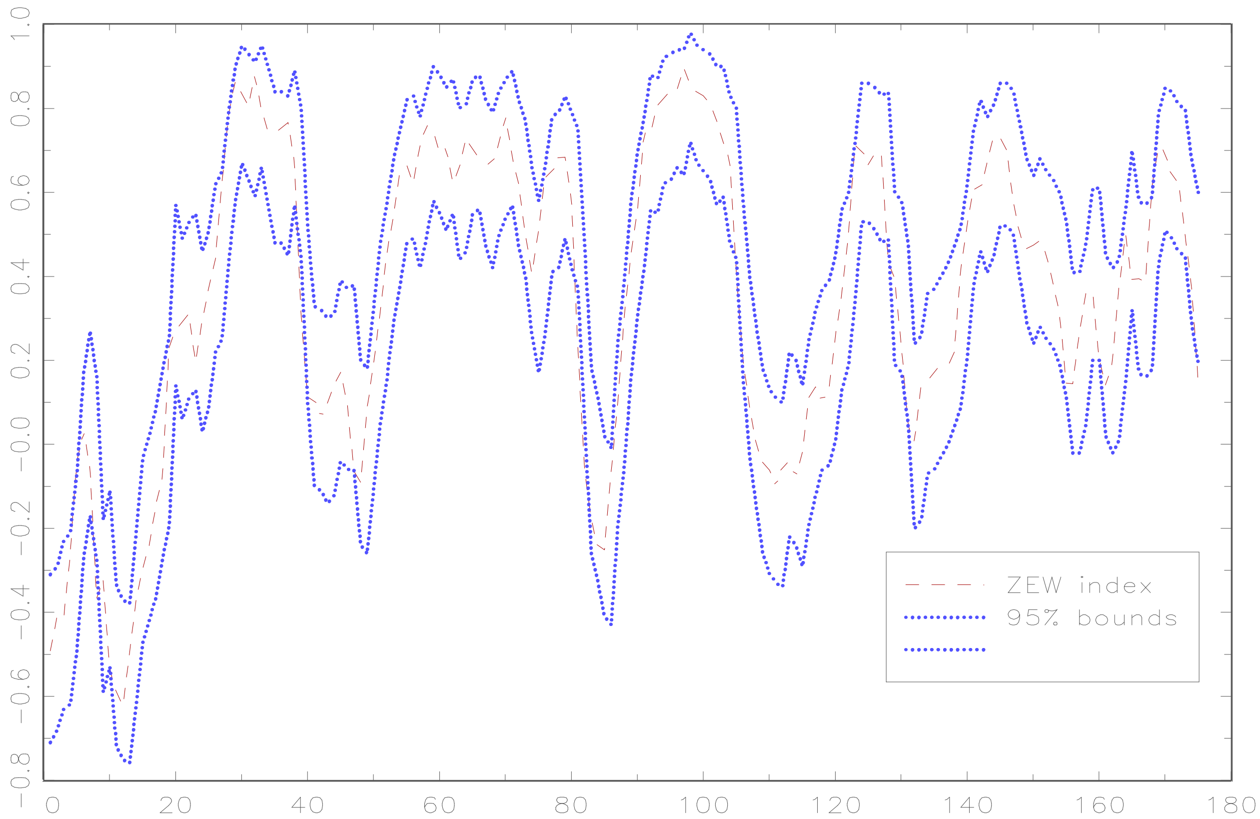


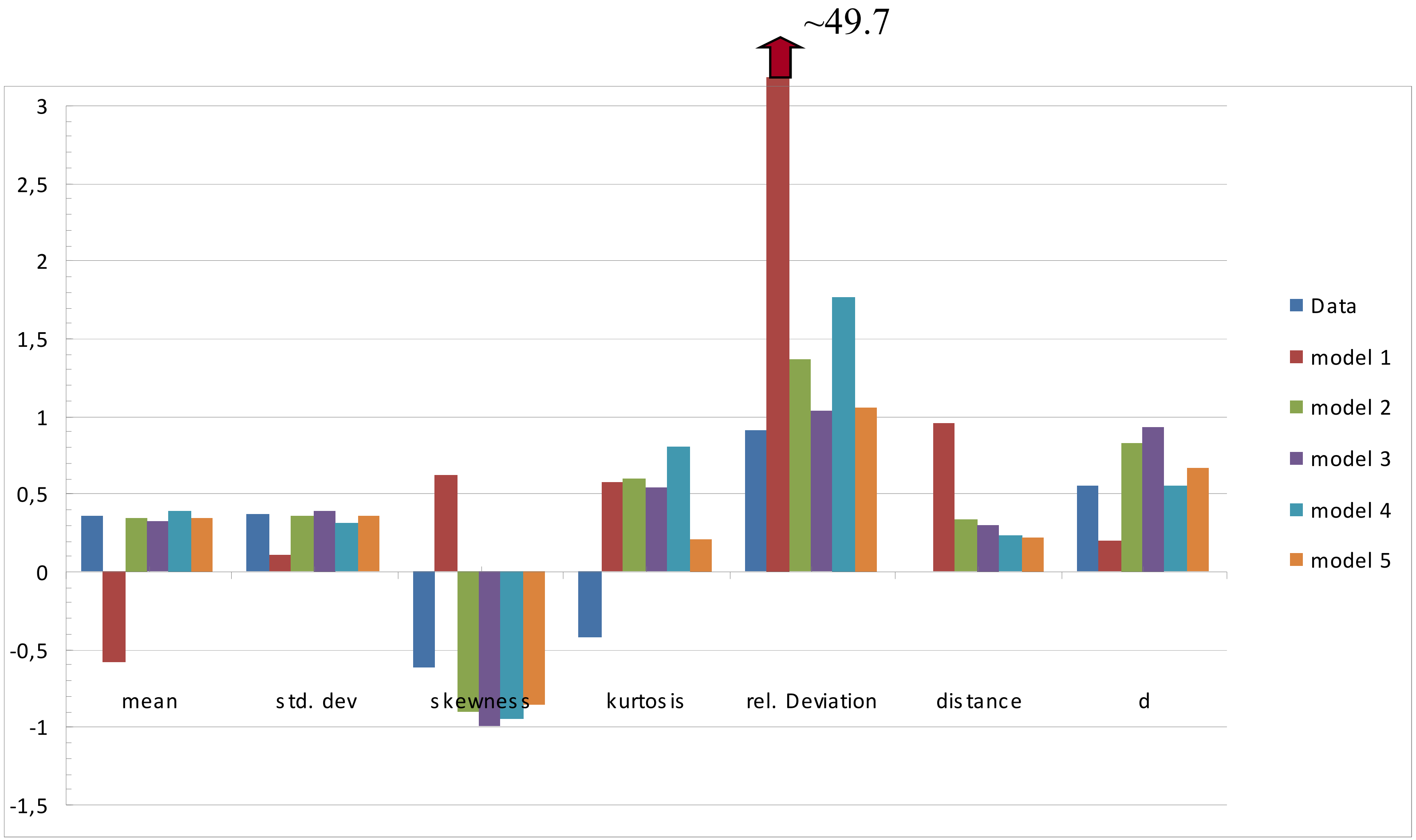
Mean and 95% confidence interval from model 1



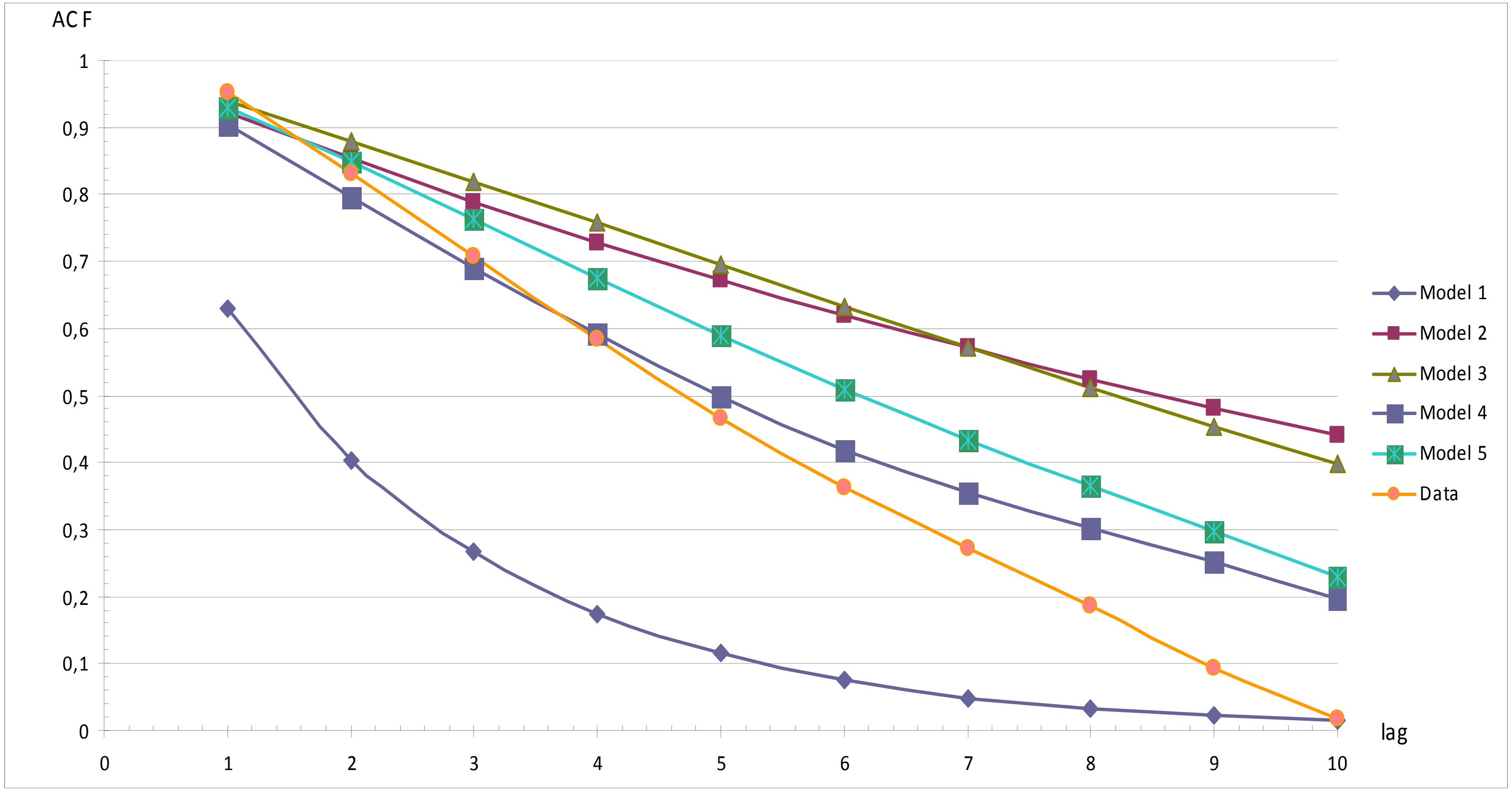
…are the large shifts of opinion in harmony with the estimated model? 95% confidence interval from period-by-period iterations (model V)

(conditional on previous realization and influence form IP)





Moments: Data vs Simulated Models (average of 1000 simulations)



Autocorrelations: Data vs Simulated Models (average of 1000 simulations)

Conclusions

- ☑ evidence for interaction effects in ZEW index ($\alpha_1 \approx 1$)
- ☑ no significant bias, slow development (v small)
- ☑ effective system size < nominal size (degree of complexity)
- ☑ some (limited) evidence of interaction with macro data
- ☑ interaction effects are dominant part of the model
- ☑ we can identify the formation of animal spirits and track their development

Avenues for further research

- other time series: in economics, finance (investor sentiment), politics, marketing

- estimating combined models with joined dynamics of opinion formation and real economic activity

- check for system size effects: correlations in individual behavior (micro data) ?

- indirect identification of psychological states from economic data