Value at Risk and Self–Similarity

by

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1 Introduction

The concept of **Value at Risk** (VaR) measures the "risk" of a portfolio. More precisely, it is a statement of the following form:

With probability $1 - q$ the potential loss will not exceed the Value at Risk–figure.

Speaking in mathematical terms, this is simply the q–quantile of the distribution of the change of value for a given portfolio P. More specifically,

$$VaR_q\left(P^d\right) = -F_{P^d}^{-1}(1 - q),$$

where P^d is the change of value for a given portfolio over d days (the d–day return) and F_{P^d} is the distribution function of P^d.

In the daily use it is common to derive the Value at Risk–figure of d days from the one of one–day by multiplying the Value at Risk–figure of one–day by $\sqrt{d}$, that is

$$\boldsymbol{VaR_q\left(P^d\right) = \sqrt{d} \cdot VaR_q\left(P^1\right)}. \tag{1}$$

Even banking supervisors recommend this procedure (see the Basel Committee on Banking Supervision).

On the other hand, if the changes of the value of the considered portfolio P are self–similar with Hurst coefficient H (with $H \neq \frac{1}{2}$), equation (1) has to be modified in the following way:

$$\boldsymbol{VaR_q\left(P^d\right) = d^H \cdot VaR_q\left(P^1\right)}. \tag{2}$$

To verify this equation, let us first recall the definition of self–similarity.

Definition 1.1

A real–valued process $(X(t))_{t\in\mathbb{R}}$ *is* **self–similar with index** $H > 0$ *(*H*–ss) if for all* $a > 0$*, the finite–dimensional distributions of* $(X(at))_{t\in\mathbb{R}}$ *are identical to the finite–dimensional distributions of* $\left(a^H X(t)\right)_{t\in\mathbb{R}}$*, i.e., if for any* $a > 0$

$$(X(at))_{t\in\mathbb{R}} \stackrel{d}{=} \left(a^H X(t)\right)_{t\in\mathbb{R}}.$$

This implies

$$\begin{aligned} F_{X(at)}(x) &= F_{a^H X(t)}(x) \quad \text{for all } a > 0 \text{ and } t \in \mathbb{R} \\ &= P\left(a^H X(t) < x\right) \\ &= P\left(X(t) < a^{-H} x\right) \\ &= F_{X(t)}\left(a^{-H} x\right). \end{aligned}$$

2 Possible Size of Risk Underestimation

Value at Risk Underestimation for Various Days
Given $H = 0.55$ and $H = 0.6$

Days d	$H = 0.55$			$H = 0.6$		
	$d^{0.55}$	$d^{0.55} - d^{\frac{1}{2}}$	Relative Difference in Percent	$d^{0.6}$	$d^{0.6} - d^{\frac{1}{2}}$	Relative Difference in Percent
1	1	0	0	1	0	0
2	1.46	0.05	3.53	1.52	0.1	7.18
5	2.42	0.19	8.38	2.63	0.39	17.46
10	**3.55**	**0.39**	**12.2**	**3.98**	**0.82**	**25.89**
30	6.49	1.02	18.54	7.7	2.22	40.51
50	8.6	1.53	21.6	10.46	3.39	47.88
100	12.59	2.59	25.89	15.85	5.85	58.49
250	**20.84**	**5.03**	**31.79**	**27.46**	**11.65**	**73.7**

This table shows d^H, the difference between d^H and $\sqrt{d}$, and the relative difference $\frac{d^H - \sqrt{d}}{\sqrt{d}}$ for various days d and for $H = 0.55$ and $H = 0.6$.

3 Estimation of the Hurst Exponent via Quantiles

There are different methods for estimating the Hurst exponent:

- the p–*th moment method* (with $p \in \mathbb{N}$), possibly together with a quantile plot (the so–called QQ–plot),
- the *R*– or the *R/S–statistics* (by Hurst or Lo) which captures only the long range dependencies, and
- the *Hill estimator* or the *peaks over threshold method (POT)*, which are both in widespread use in the extreme value theory (and which does not account for long range dependencies),

just to mention the most popular ones.

3.1 Theoretical Foundations

In order to derive an alternative estimation of the Hurst exponent, let us recall, that

$$VaR_q\left(P^d\right) = d^H \cdot VaR_q\left(P^1\right) ,$$

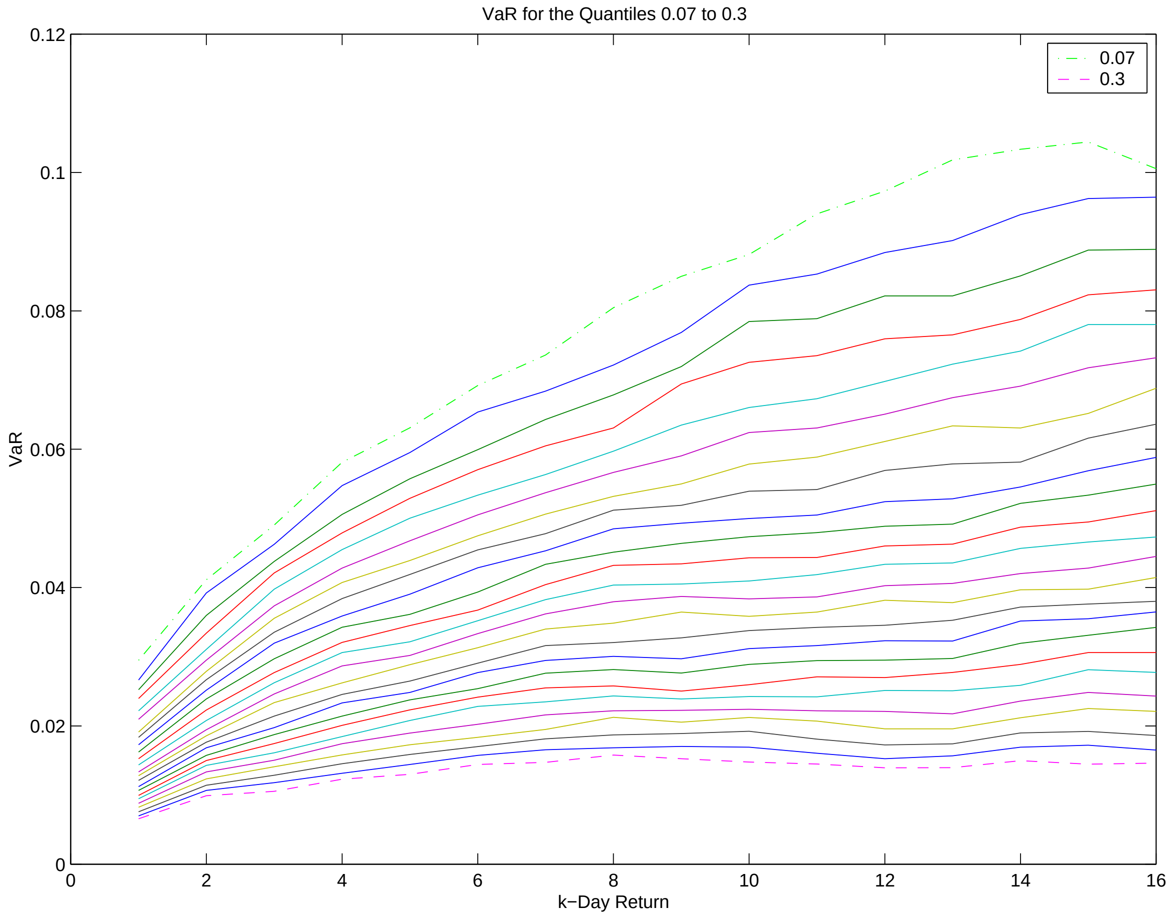
if $\left(P^d\right)$ is H–ss. Given this, it is easy to derive that

$$\boldsymbol{\log\left(VaR_q\left(P^d\right)\right) = H \cdot \log\left(d\right) + \log\left(VaR_q\left(P^1\right)\right).} \quad (3)$$

Thus the Hurst exponent can be derived from the gradient of a linear regression in a log–log–plot. In particular,

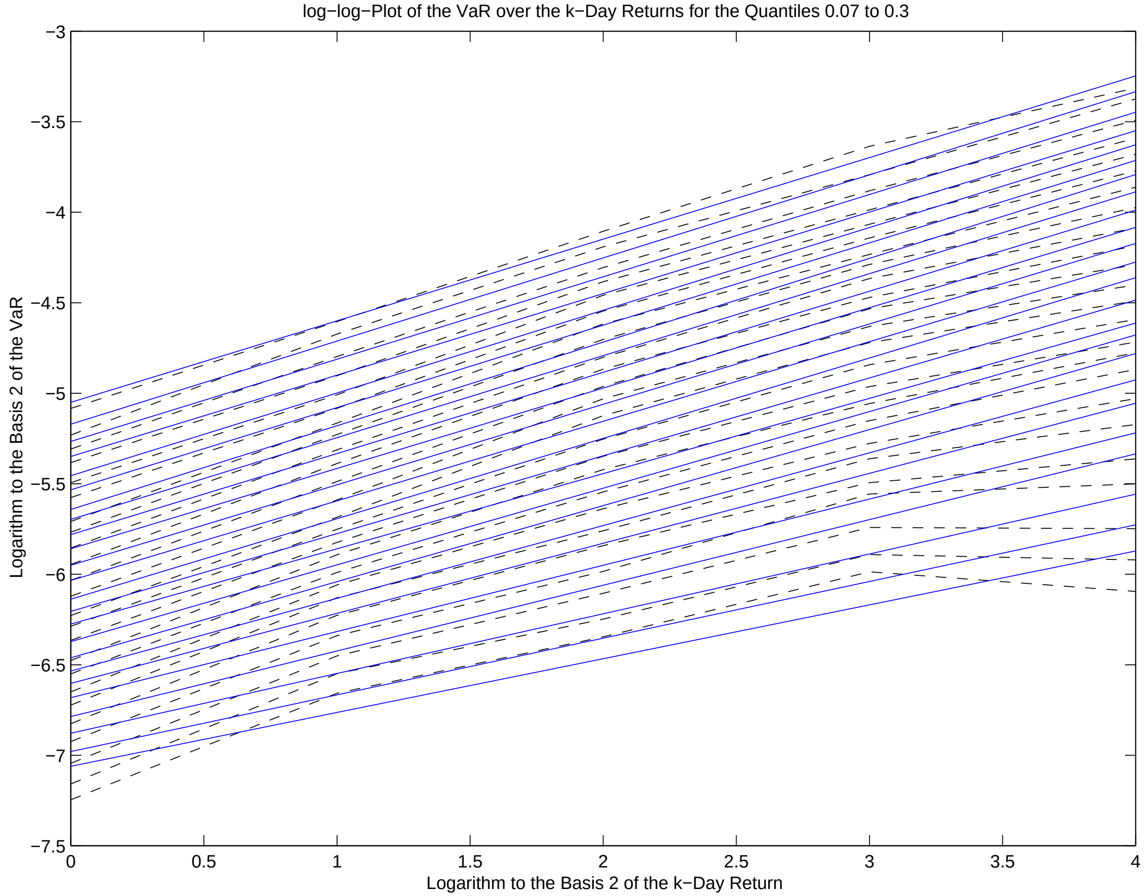
- it is not necessary that the higher moments of the underlying process exist for this approach.
- it is possible to observe the evolution of the estimation of the Hurst coefficient along the various quantiles.

Various Value at Risk Figures for the SAP Daily Closing Stock Prices, based on Commercial Return



This graphic shows various Value at Risk figures based on the daily closing stock prices for the SAP–stock, beginning on June 18$^{\text{th}}$, 1990 and ending on January 13$^{\text{th}}$, 2000. The Value at Risk has been calculated for the k–day commercial return (with $k = 1, \ldots, 16$) and for various quantiles, ranging from 7 percent to 30 percent.

Log–Log–plot for the SAP–Stock, Based on Commercial Return



This graphic shows the logarithm to the basis 2 of various Value at Risk figures based on the daily closing stock prices for the SAP–stock, beginning on June 18$^{\text{th}}$, 1990 and ending on January 13$^{\text{th}}$, 2000. The Value at Risk has been calculated for the 2^j–day commercial return (with $j = 0, \ldots, 4$) and for various quantiles, ranging from 7 percent to 30 percent (the black dashed lines) from top to bottom, respectively. The solid blue lines show the linear regression for the various quantiles.

There are several approaches for calculating the Value at Risk–figure. The most popular are the

- variance–covariance approach,
- **historical simulation**,
- Monte–Carlo simulation, and
- extreme value theory.

In practice banks often estimate the Value at Risk via **order statistics**, which is the focus of this talk. Let $G_{j:n}(x)$ be the distribution function of the j–th order statistics. Since the probability, that exactly j observations (of a total of n observations) are less or equal x, is given by

$$\frac{n!}{j! \cdot (n-j)!} F(x)^j \left(1 - F(x)\right)^{n-j},$$

it can be verified, that

$$G_{j:n}(x) = \sum_{k=j}^{n} \frac{n!}{k! \cdot (n-k)!} F(x)^k \left(1 - F(x)\right)^{n-k}. \tag{4}$$

This is the probability, that at least j observations are less or equal x given a total of n observations.

Equation (4) implies, that the self–similarity holds also for the distribution function of the j–th order statistics of a self–similar random variable. In this case, one has

$$G_{j:n,P^d}(x) = G_{j:n,P^1}\left(d^{-H} \cdot x\right) .$$

It is important, that one has n observations for $\left(P^1\right)$ as well as for $\left(P^d\right)$, otherwise the equation does not hold.

This shows that the j–th order statistics preserves – and therefore shows – the self–similarity of a self–similar process. Thus the j–th order statistics can be used to estimate the Hurst exponent as it will be done in this talk.

3.2 Error of the Quantile Estimation

Obviously, (3) can only be applied, if $VaR_q\left(P^d\right) \neq 0$. Moreover, close to zero, a possible error in the quantile estimation will lead to an error in (3), which is much larger than the original error from the quantile estimation.

Let l be the number, which represents the q–th quantile of the order statistics with n observations. With this, x_l is the q–th quantile of a given time series X, which consists of n observations and with $q = \frac{l}{n}$. Let X be a stochastic process with a differentiable density function $f > 0$. Moreover the variance of x_l is

$$\sigma_{x_l}^2 = \frac{q \cdot (1-q)}{n \cdot (f(x_l))^2},$$

where f is the density function of X and f must be strictly greater than zero.

The **propagation of errors** are calculated by the total differential. Thus, the propagation of this error in (3) is given by

$$\begin{aligned} \sigma_{\log(x_l)} &= \frac{1}{x_l} \cdot \left(\frac{q \cdot (1-q)}{n \cdot (f(x_l))^2} \right)^{\frac{1}{2}} \\ &= \sqrt{\frac{q \cdot (1-q)}{n}} \cdot \frac{1}{x_l \cdot f(x_l)} \,. \end{aligned}$$

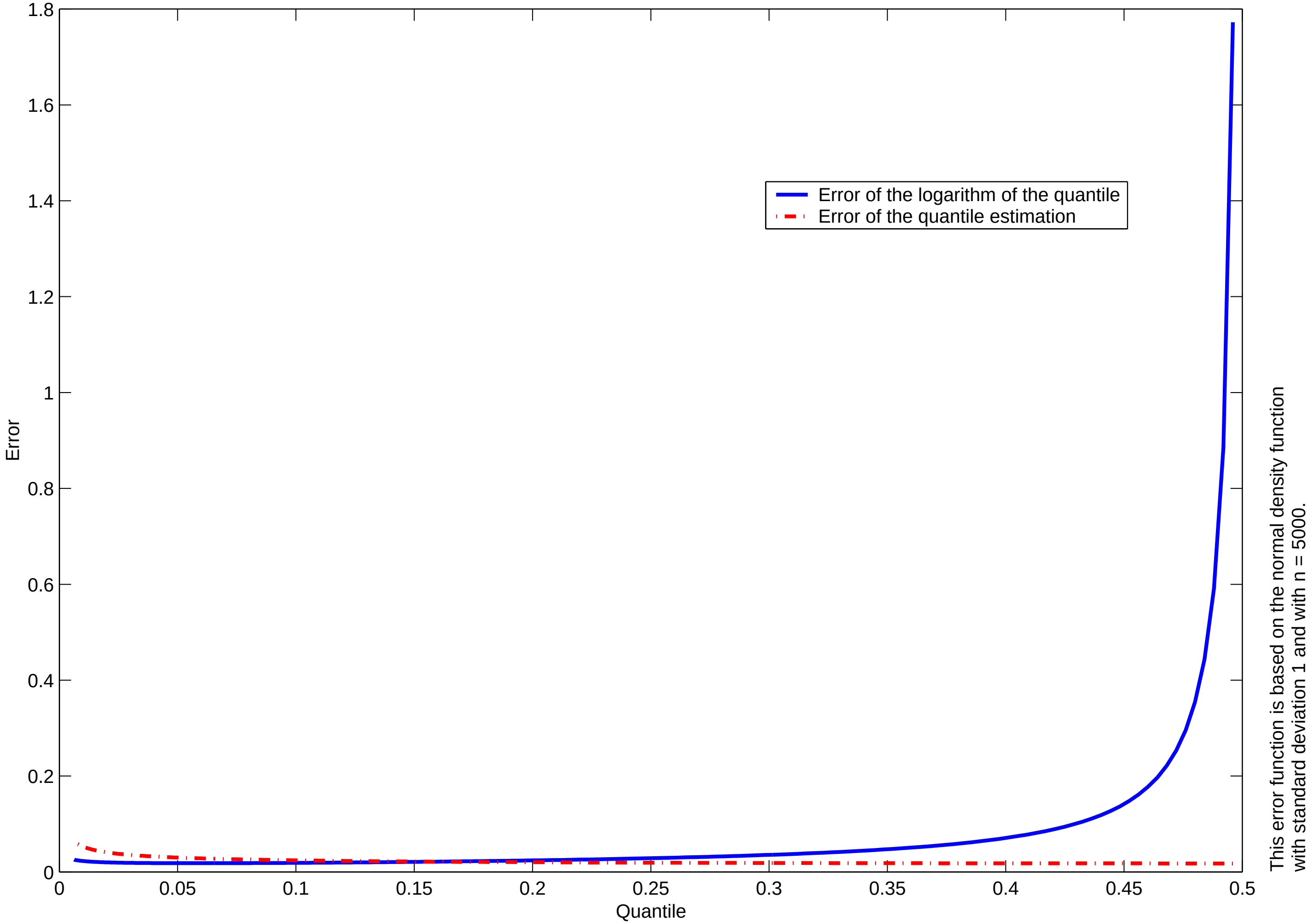
1. Example: **The normal distribution**

For example, if $X \sim \mathcal{N}(0, \sigma^2)$ the propagation of the error can be written as

$$\begin{aligned} \sigma_{\log(x)} &= \sqrt{\frac{q \cdot (1-q)}{n}} \cdot \frac{\sqrt{2\pi} \cdot \sigma}{x \cdot \exp\left(-\frac{x^2}{2\sigma^2}\right)} \\ &= \sqrt{\frac{q \cdot (1-q)}{n}} \cdot \frac{\sqrt{2\pi}}{y \cdot \exp\left(-\frac{y^2}{2}\right)}, \end{aligned}$$

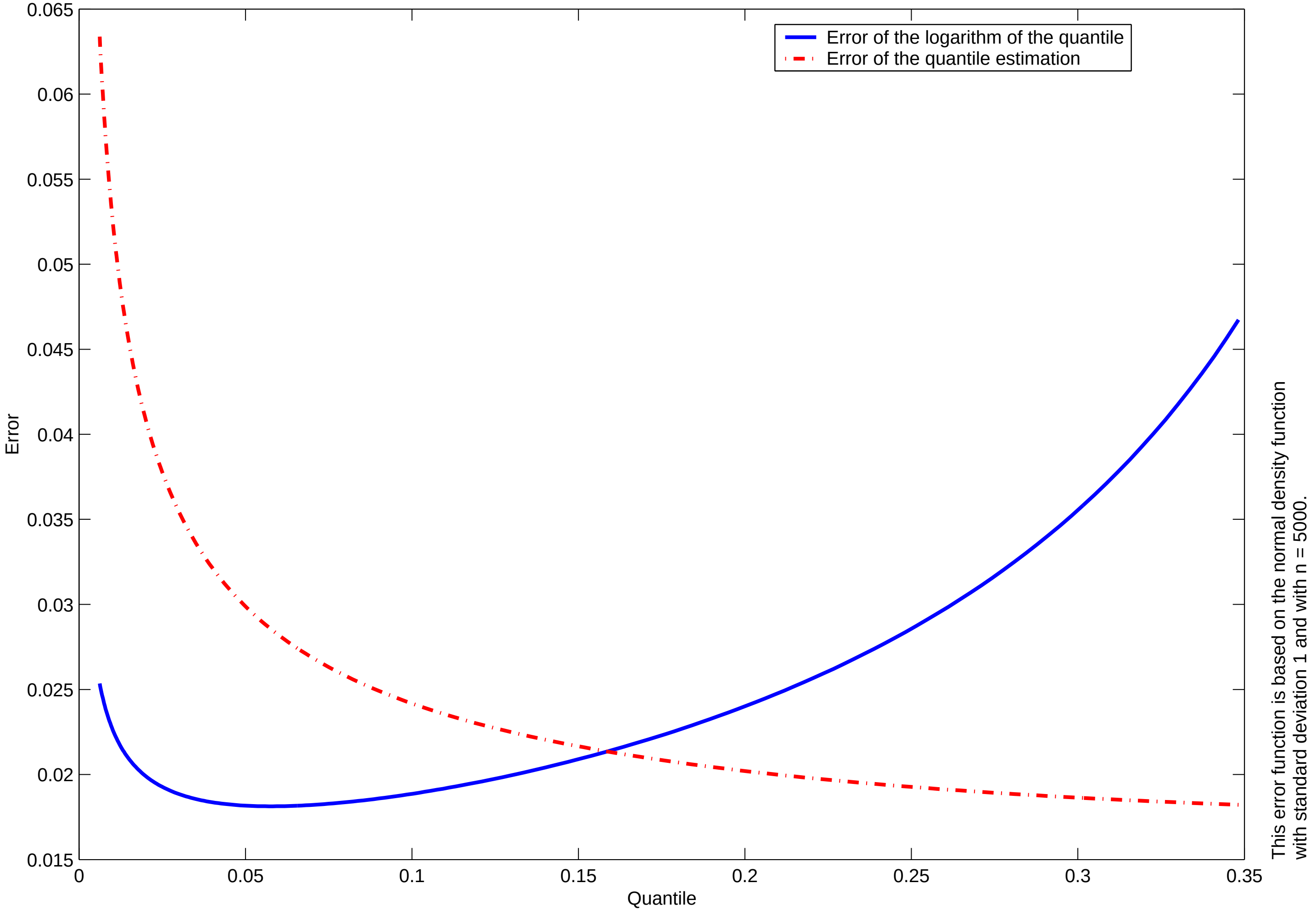
where the substitution $\sigma \cdot y = x$ has been used. This shows, that the error is independent of the variance of the underlying process, if this underlying process is normal distributed.

Error Function for the Normal Distribution



This figure shows the error curves of the quantile estimation (red dash–dotted line) and of the logarithm of the quantile estimation (blue solid line) for the normal distribution.

Error Function for the Normal Distribution, a Zoom In



This figure shows the error curves of the quantile estimation (red dash–dotted line) and of the logarithm of the quantile estimation (blue solid line) for the normal distribution.

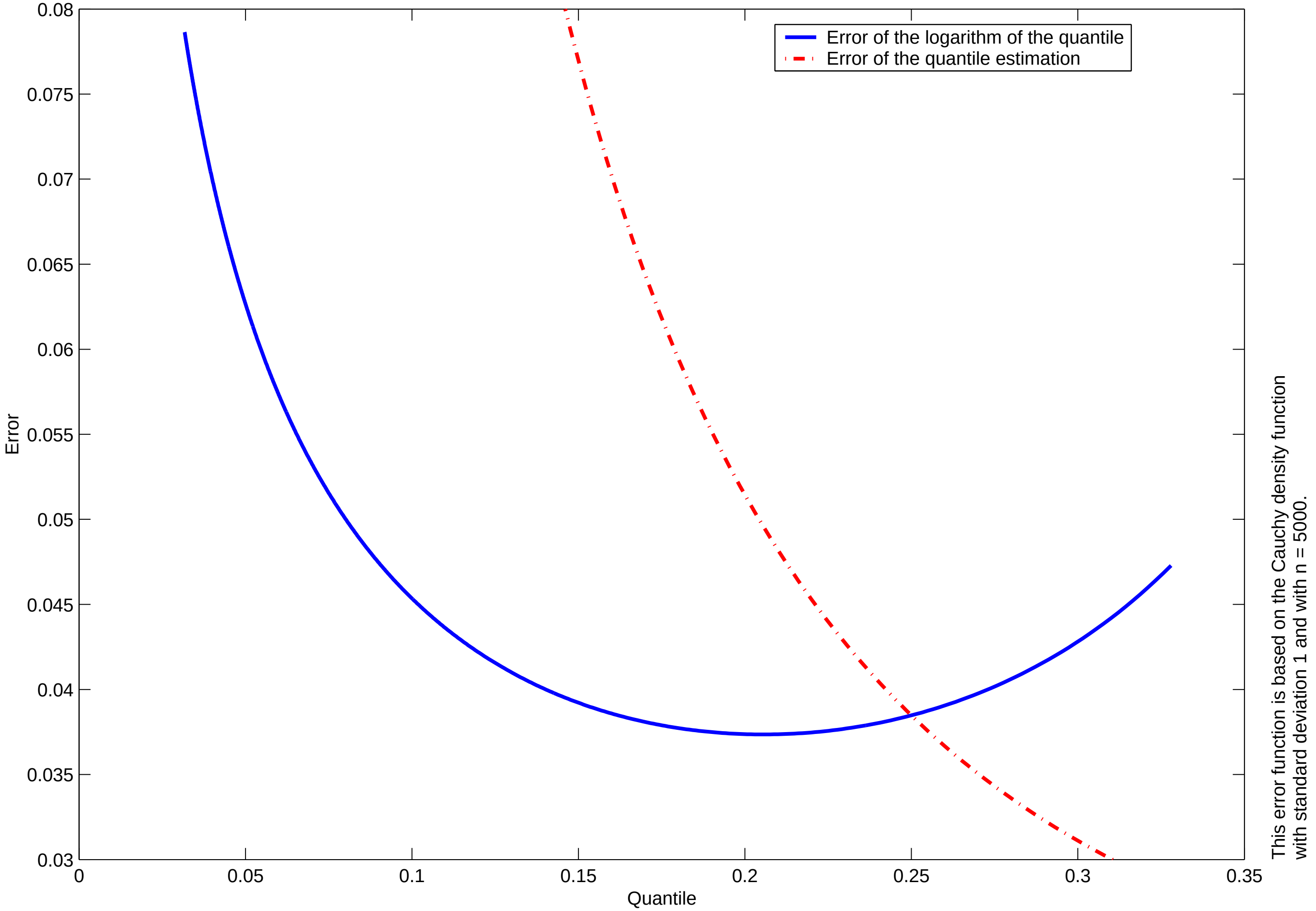
2. Example: **The Cauchy distribution**

Similarly, if X is Cauchy with mean zero, the propagation of the error can be shown to be

$$\sigma_{\log(x)} = \sqrt{\frac{q\cdot(1-q)}{n}}\cdot\frac{\pi\cdot(x^2+\sigma^2)}{\sigma\cdot x} = \sqrt{\frac{q\cdot(1-q)}{n}}\cdot\frac{\pi\cdot(y^2+1)}{y},$$

where the substitution $\sigma\cdot y = x$ has also been used.

Error Function for the Cauchy Distribution, a Zoom In



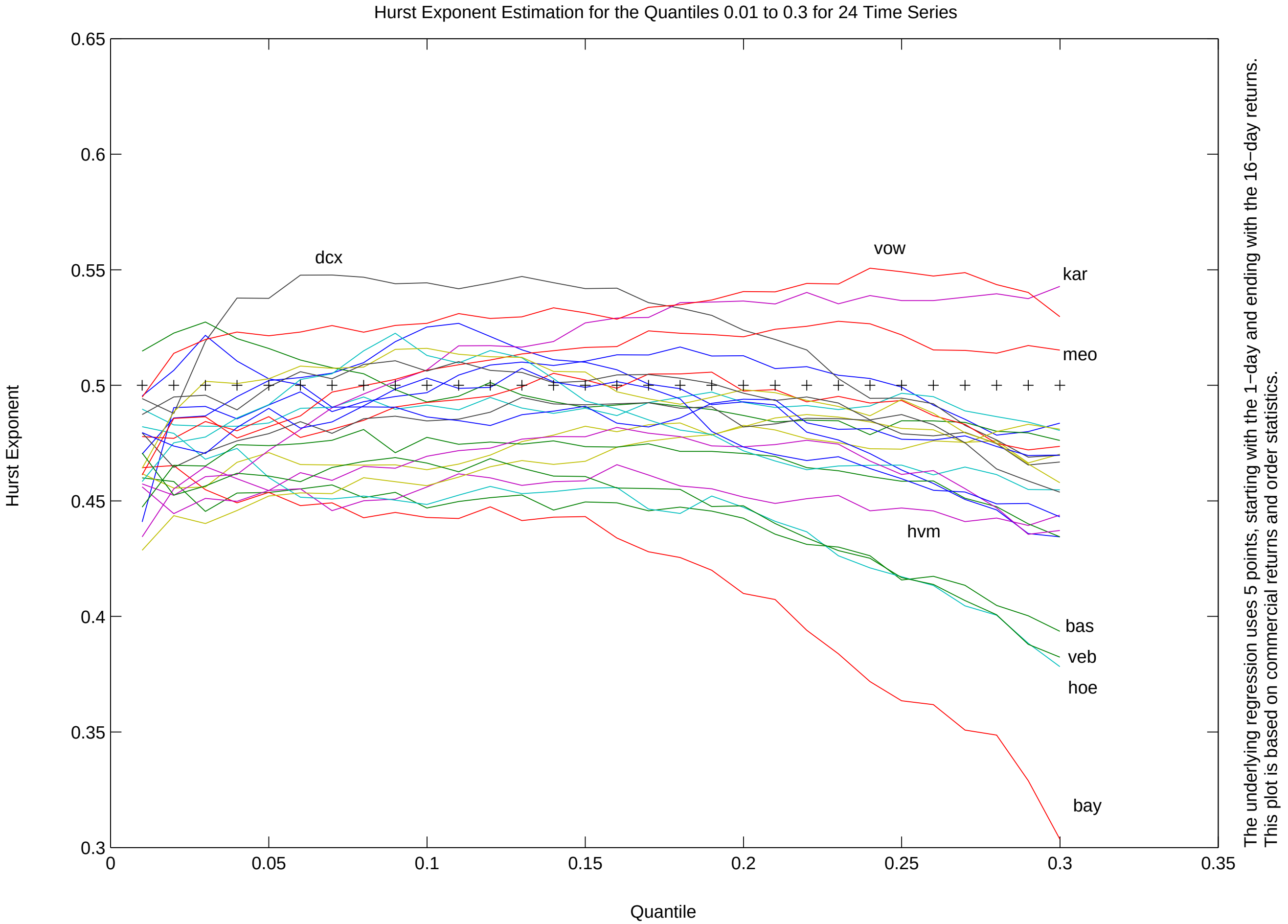
This figure shows the error curves of the quantile estimation (red dash–dotted line) and of the logarithm of the quantile estimation (blue solid line) for the Cauchy distribution.

4 Estimating the Scaling Law for Some Stocks

A self–similar process with Hurst exponent H can not have a drift. Since it is recognized that financial time series do have a drift, they can not be self–similar. Because of this the wording **scaling law** instead of *Hurst exponent* will be used when talking about financial time series, which have *not* been detrended.

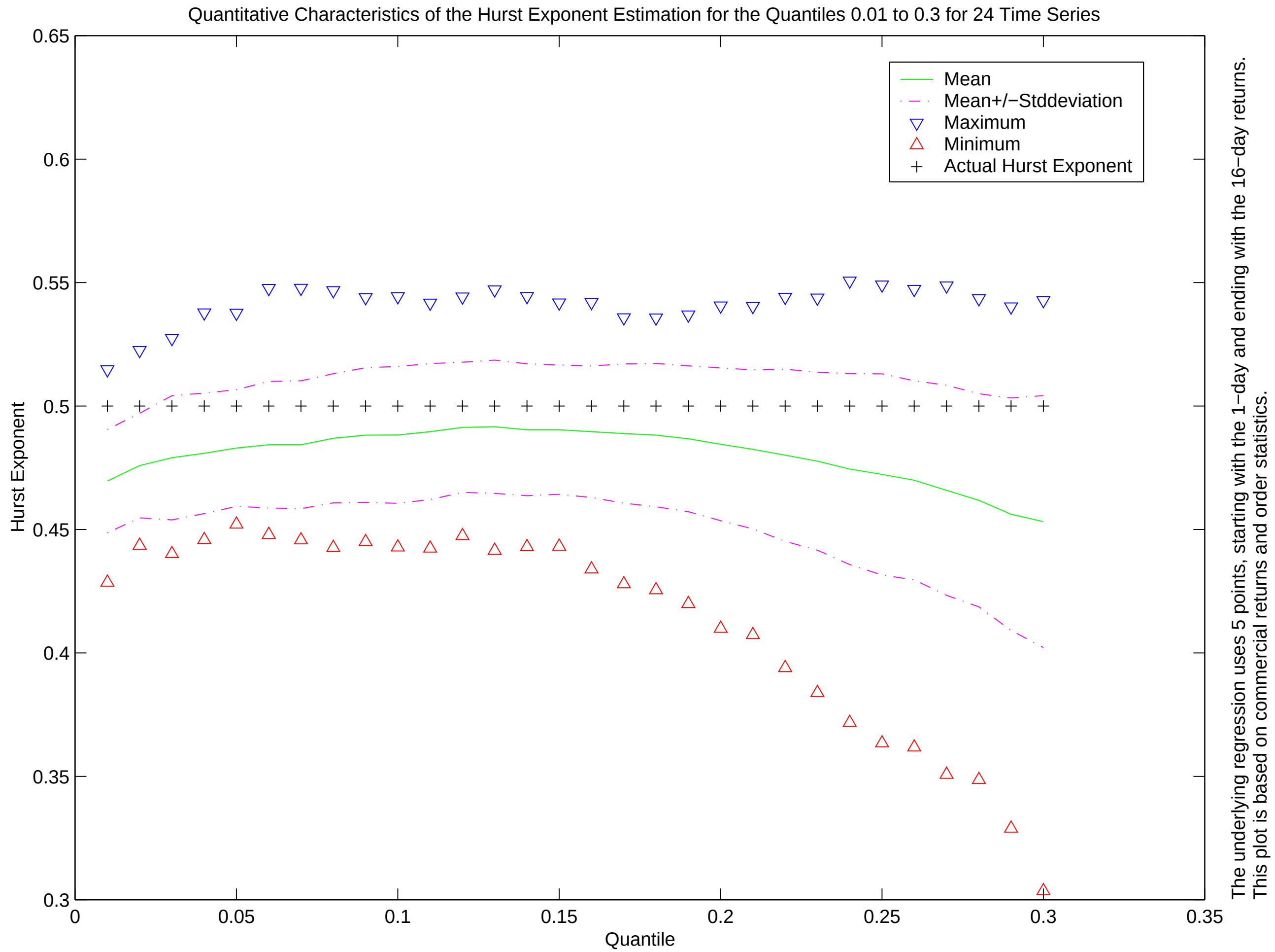
Since the scaling law is more relevant in practice than in theory,only those figures are depicted which are based on commercial returns.

Estimation of the Scaling Law for 24 DAX–Stocks, Left Side



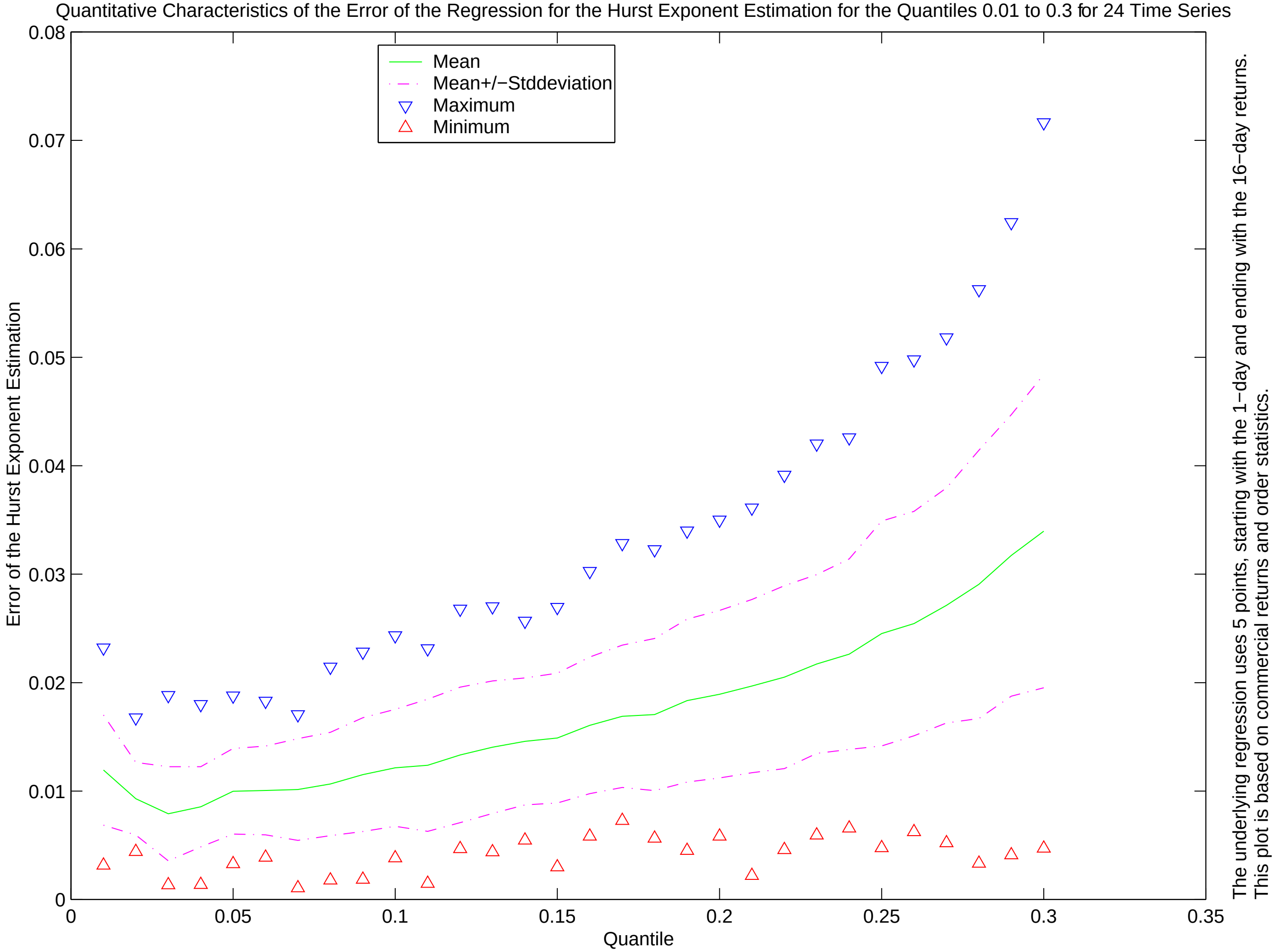
Shown are the estimation for the scaling law for 24 DAX–stocks. The underlying time series is a commercial return. Shown are the lower (left) quantiles. The following stocks are denoted explicitly: DaimlerChysler (dcx), Karstadt (kar), Volkswagen (vow), Metro (meo), Hypovereinsbank (hvm), BASF (bas), Veba (veb), Hoechst (hoe), and Bayer (bay).

Estimation of the Scaling Law for 24 DAX–Stocks, Left Side



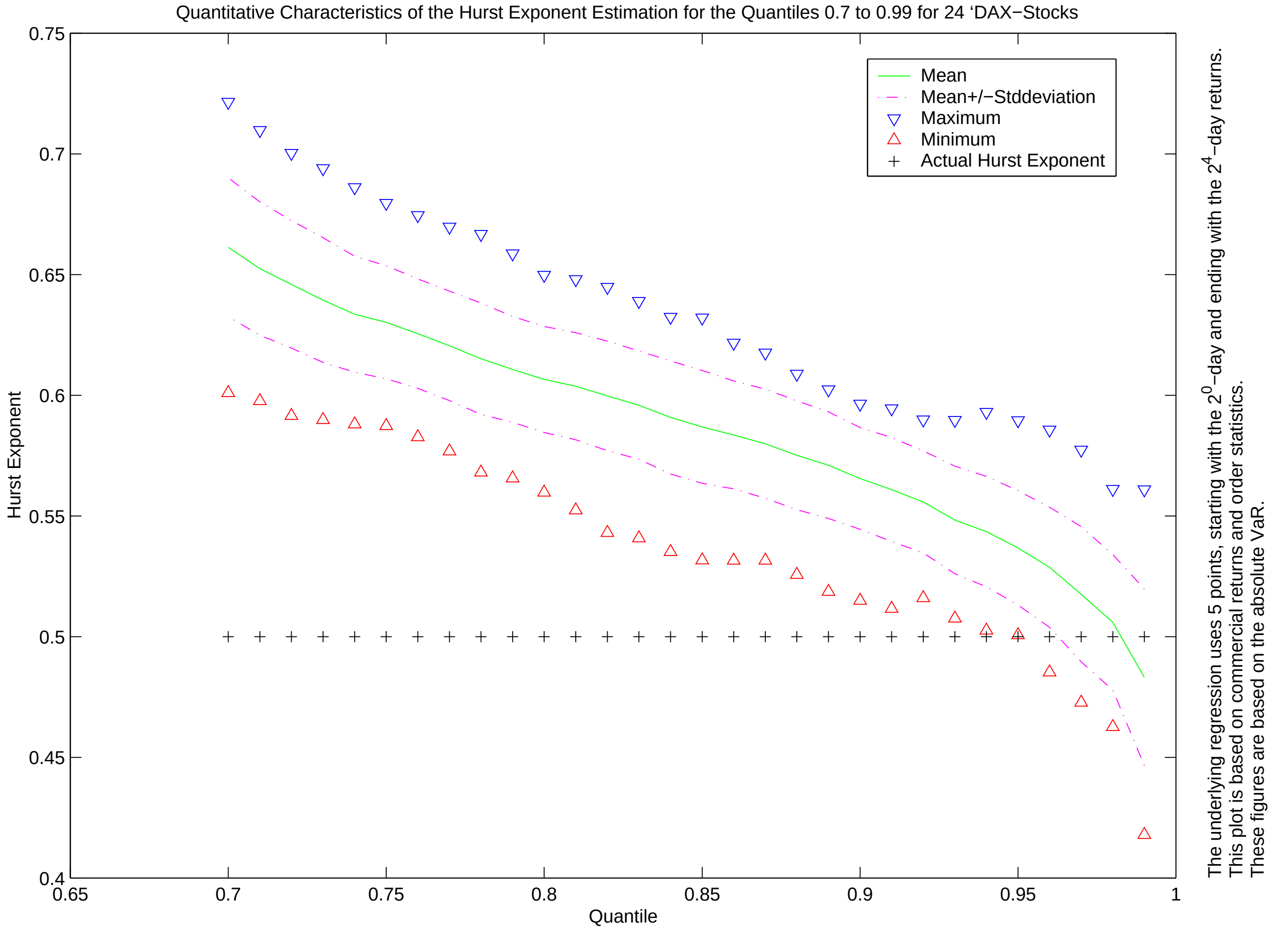
The green solid line is the mean, the magenta dash–dotted lines are the mean plus/minus the standard deviation, the red triangles are the minimum, and the blue upside down triangles are the maximum of the estimation for the scaling law, which are based on 24 DAX–stocks. The underlying time series is a commercial return. Shown are the lower (left) quantiles.

Error of the Estimation of the Scaling Law for 24 DAX–Stocks



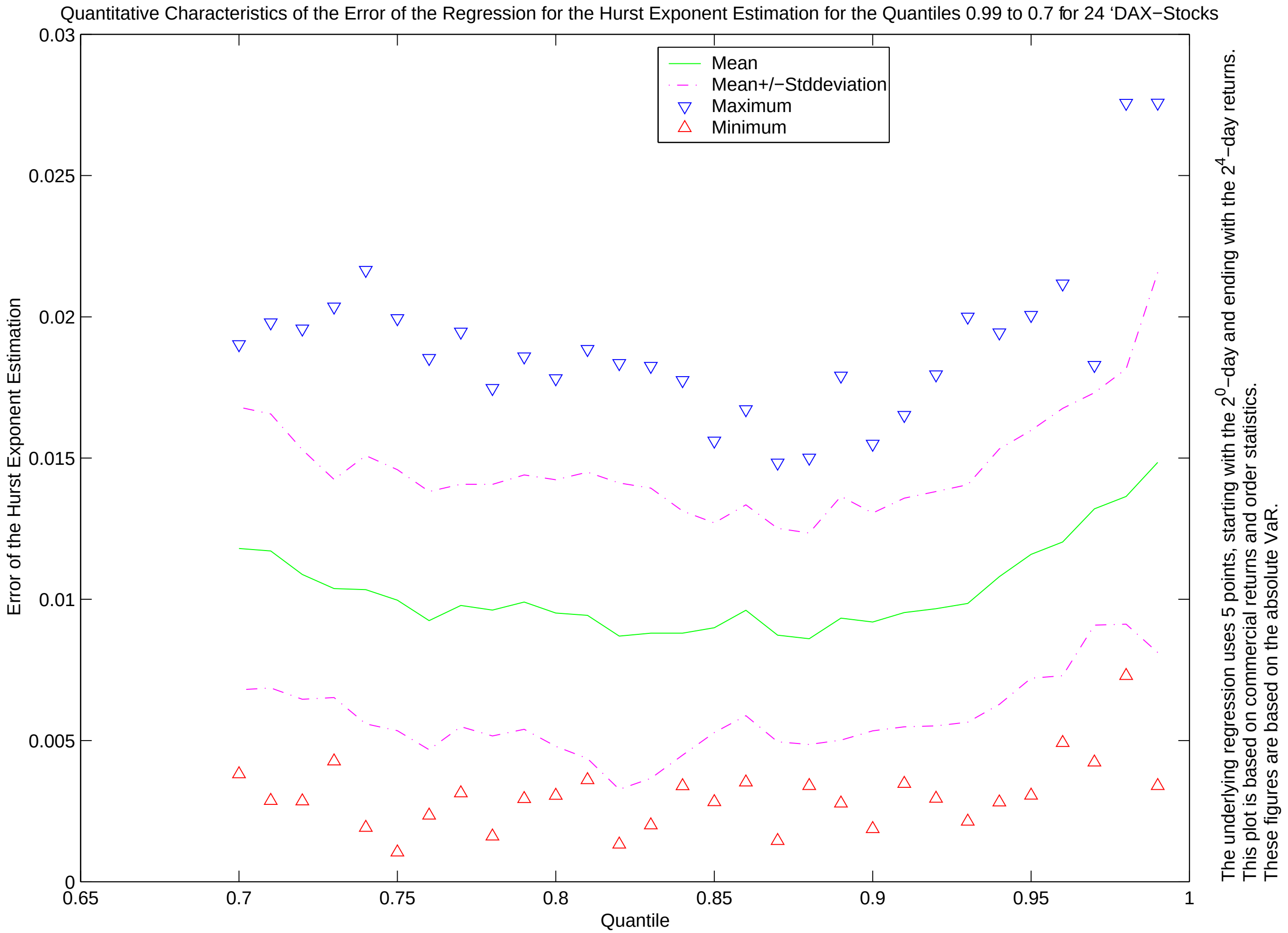
The green solid line is the mean, the magenta dash–dotted lines are the mean plus/minus the standard deviation, the red triangles are the minimum, and the blue upside down triangles are the maximum of the error curves of the estimation for the scaling law, which are based on 24 DAX–stocks. The underlying time series is a commercial return.

Estimation of the Scaling Law for 24 DAX–Stocks, Right Side



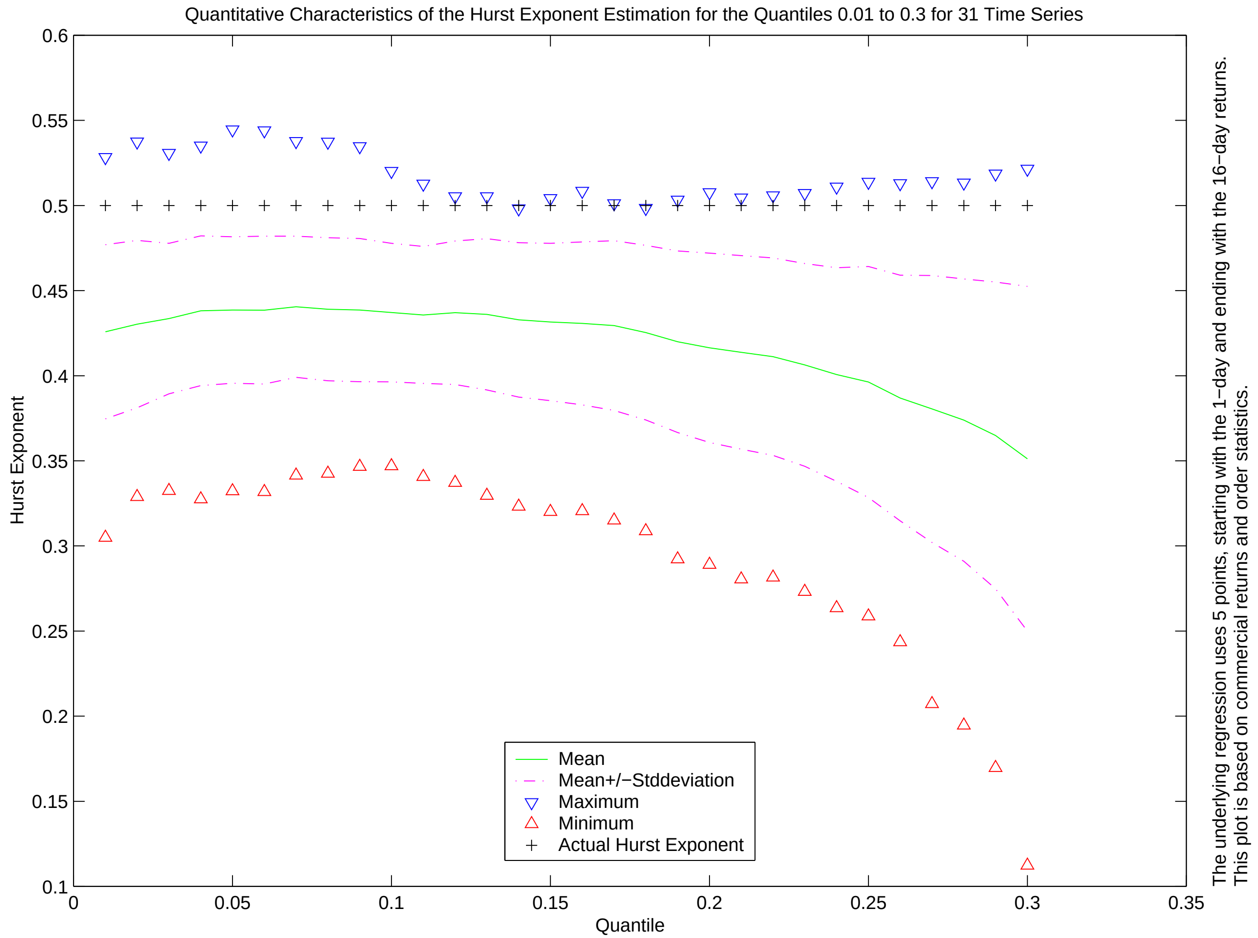
The green solid line is the mean, the magenta dash–dotted lines are the mean plus/minus the standard deviation, the red triangles are the minimum, and the blue upside down triangles are the maximum of the estimation for the scaling law, which are based on 24 DAX–stocks. The underlying time series is a commercial return. Shown are the upper (right) quantiles.

Error of the Estimation of the Scaling Law for 24 DAX–Stocks



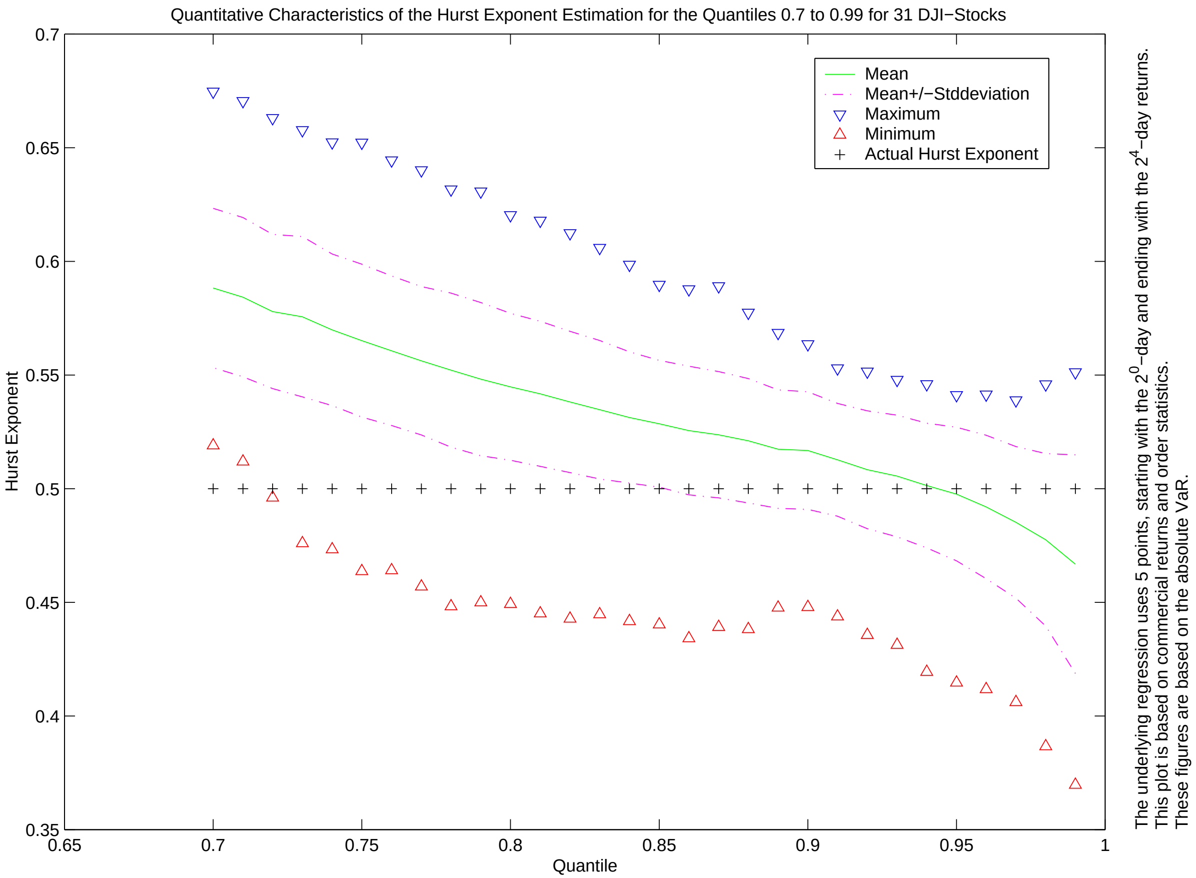
The green solid line is the mean, the magenta dash–dotted lines are the mean plus/minus the standard deviation, the red triangles are the minimum, and the blue upside down triangles are the maximum of the error curves of the estimation for the scaling law, which are based on 24 DAX–stocks. The underlying time series is a commercial return.

Estimation of the Scaling Law for the DJI and its Stocks, LQ



The green solid line is the mean, the magenta dash–dotted lines are the mean plus/minus the standard deviation, the red triangles are the minimum, and the blue upside down triangles are the maximum of the estimation for the scaling law, which are based on the DJI and its 30 stocks. The underlying time series is a commercial return. Shown are the lower (left) quantiles.

Estimation of the Scaling Law for the DJI and its Stocks, RQ



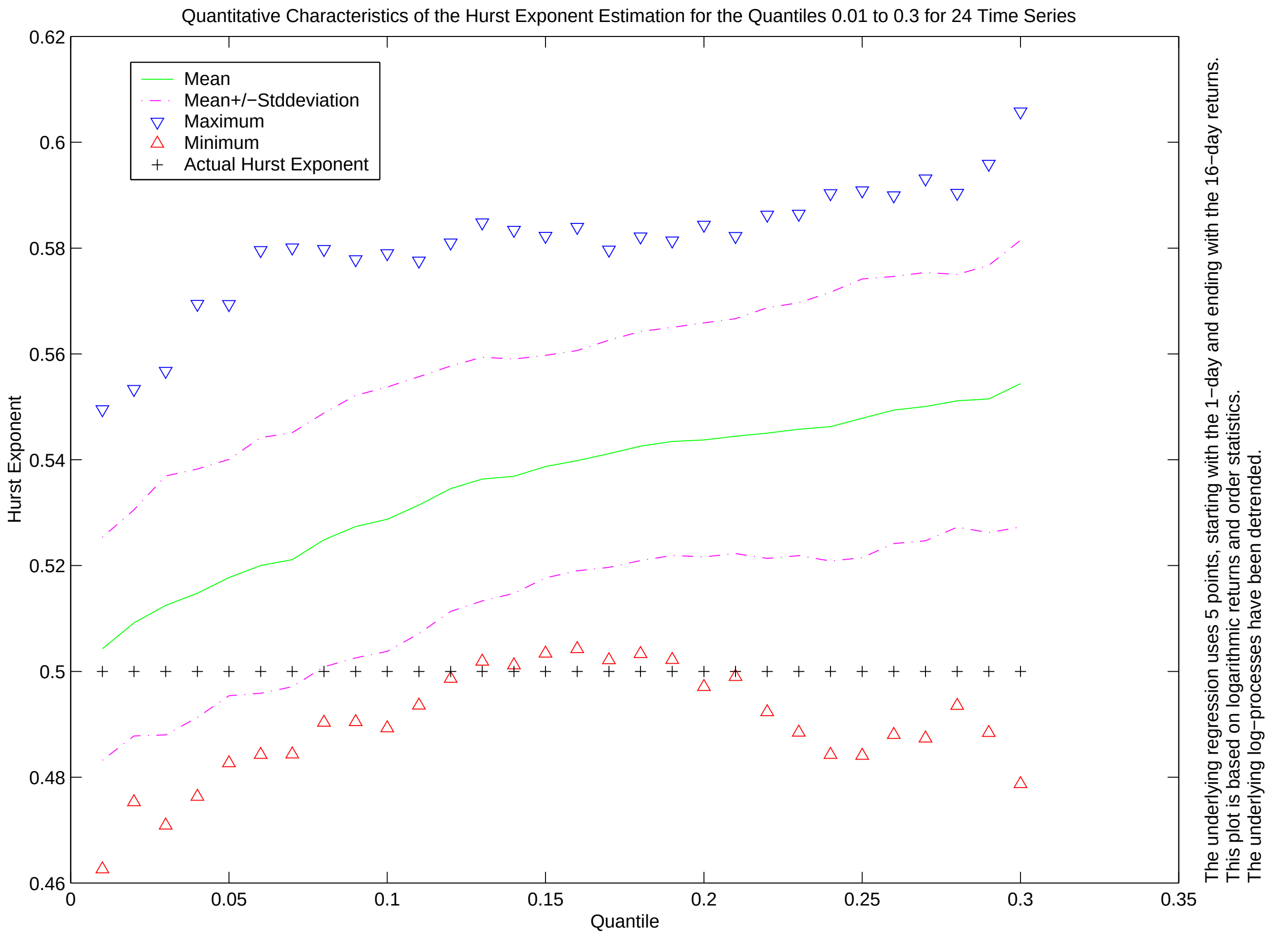
The green solid line is the mean, the magenta dash–dotted lines are the mean plus/minus the standard deviation, the red triangles are the minimum, and the blue upside down triangles are the maximum of the estimation for the scaling law, which are based on the DJI and its 30 stocks. The underlying time series is a commercial return. Shown are the upper (right) quantiles.

5 Determining the Hurst Exponent

It has been already stated, that the financial time series can not be self–similar. However, it is possible that the detrended financial time series are self–similar with Hurst exponent H. This will be scrutinized in the following where the financial time series have been detrended.

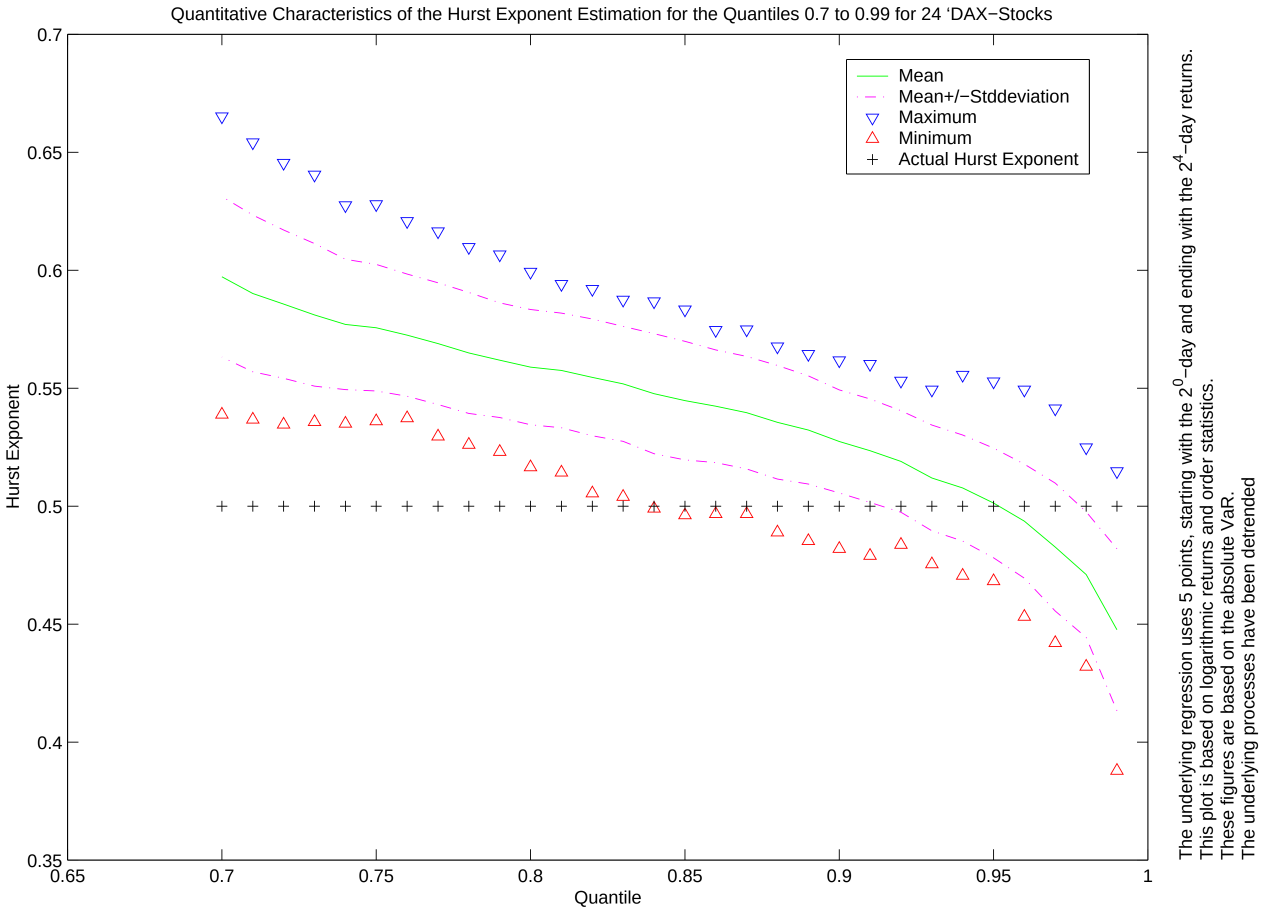
Since the Hurst exponent is more relevant in theory than in practice, only those figures are shown which are based on logarithmic returns.

Hurst Exponent Estimation for 24 DAX–Stocks, Left Quantile



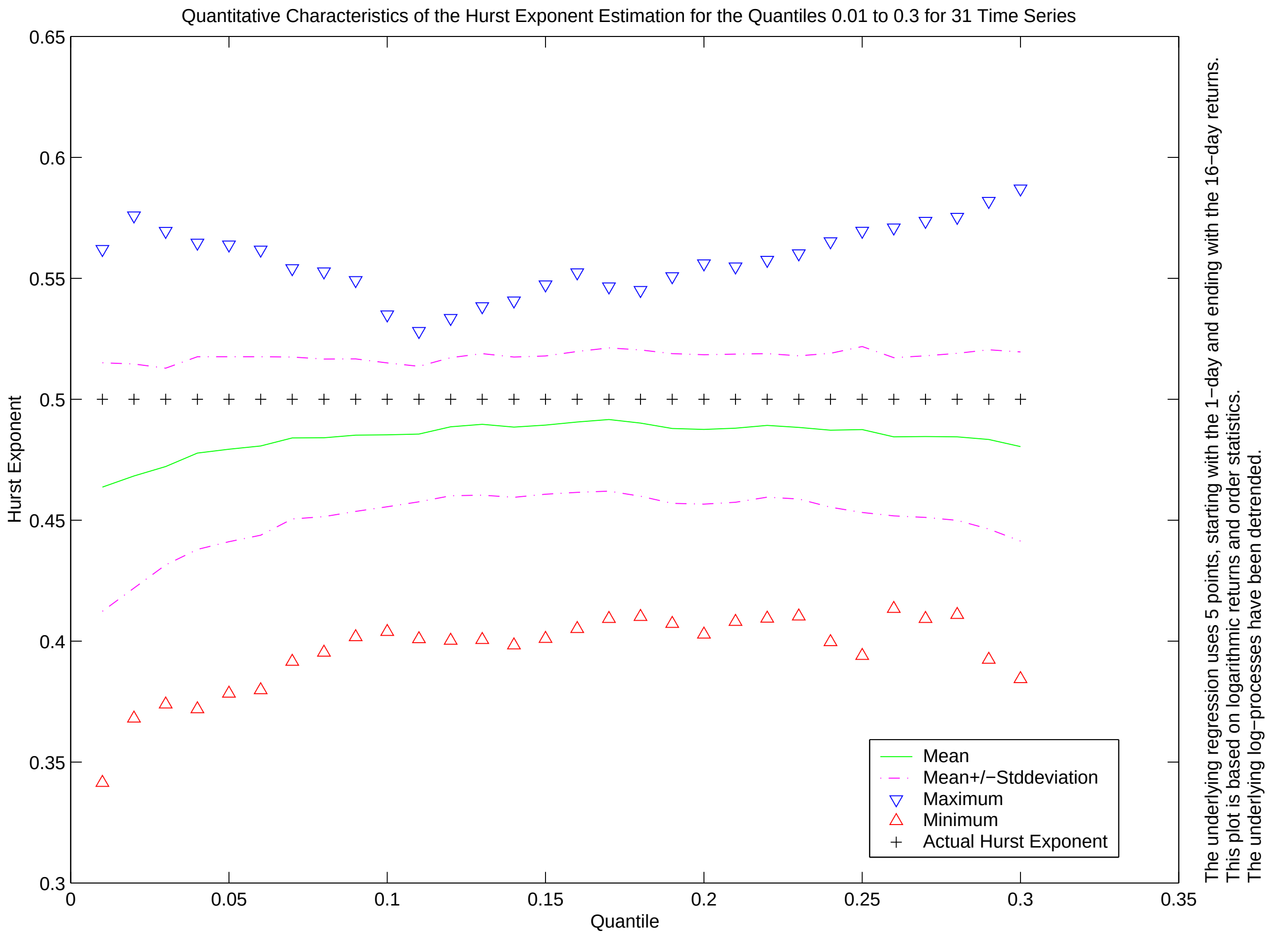
The green solid line is the mean, the magenta dash–dotted lines are the mean plus/minus the standard deviation, the red triangles are the minimum, and the blue upside down triangles are the maximum of the estimation for the Hurst exponent, which are based on 24 DAX–stocks. The underlying time series are logarithmic returns, which have been detrended. Shown are the lower (left) quantiles.

Hurst Exponent Estimation for 24 DAX–Stocks, Right Quantile



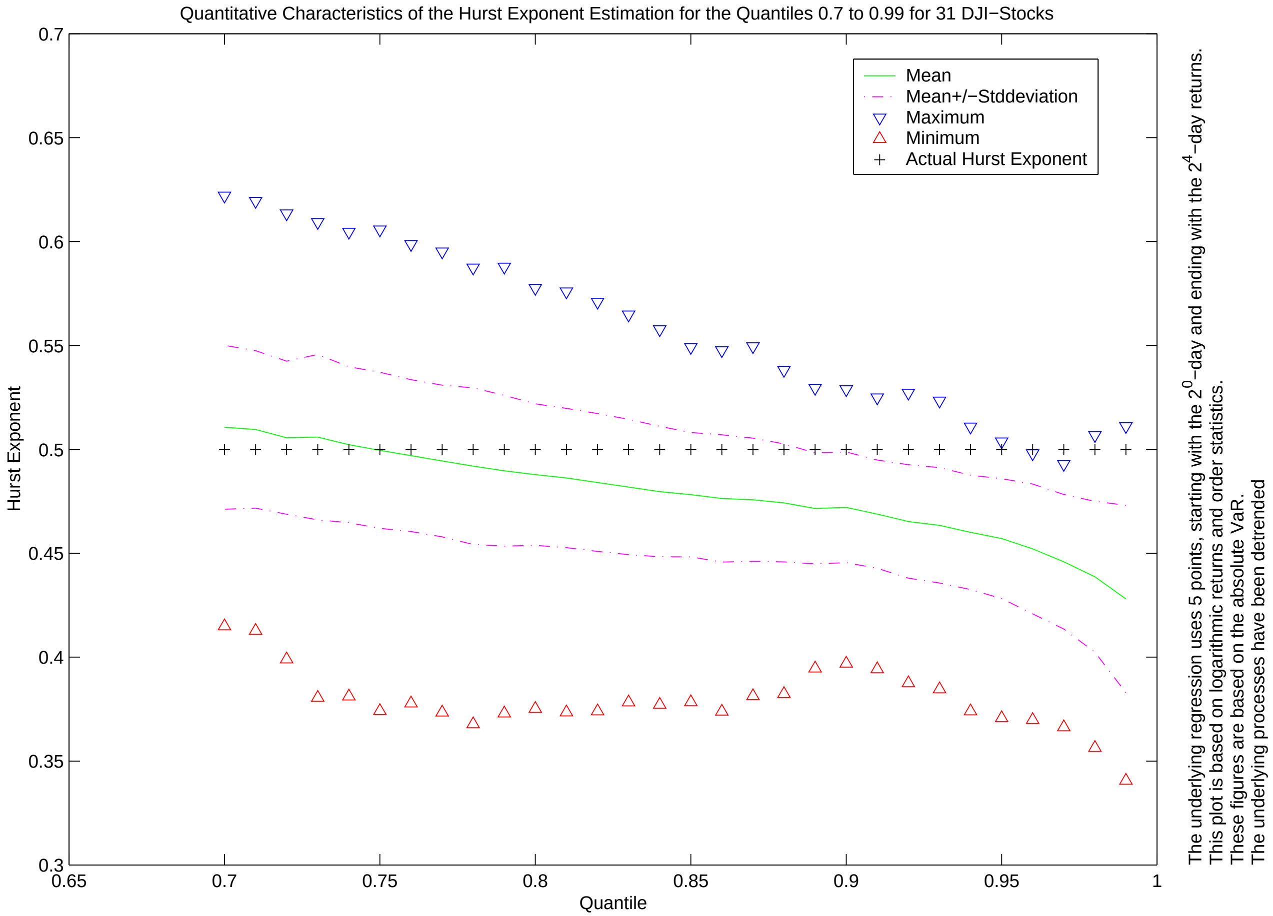
The green solid line is the mean, the magenta dash–dotted lines are the mean plus/minus the standard deviation, the red triangles are the minimum, and the blue upside down triangles are the maximum of the estimation for the Hurst exponent, which are based on 24 DAX–stocks. The underlying time series are logarithmic returns, which have been detrended. Shown are the upper (right) quantiles.

Hurst Exponent Estimation for the DJI and its Stocks, LQ



The green solid line is the mean, the magenta dash–dotted lines are the mean plus/minus the standard deviation, the red triangles are the minimum, and the blue upside down triangles are the maximum of the estimation for the Hurst exponent, which are based on the DJI and its 30 stocks. The underlying time series are logarithmic returns, which have been detrended. Shown are the lower (left) quantiles.

Hurst Exponent Estimation for the DJI and its Stocks, RQ



The green solid line is the mean, the magenta dash–dotted lines are the mean plus/minus the standard deviation, the red triangles are the minimum, and the blue upside down triangles are the maximum of the estimation for the Hurst exponent, which are based on the DJI and its 30 stocks. The underlying time series are logarithmic returns, which have been detrended. Shown are the upper (right) quantiles.

6 Interpretation of the Hurst Exponent for Financial Time Series

There are different possibilities of which we discuss the two most relevant ones:

- Fractional Brownian Motion:
  H describes the persistence of the process.

  $0 < H < \frac{1}{2}$ means that the process is anti–persistent.
  $\frac{1}{2} < H < 1$ means that the process is persistent.

- α–stable Lévy Process:
  $H = \frac{1}{\alpha}$ for $\alpha \in [1, 2]$ determines how much the process is heavy–tailed.

The bigger H is, the more heavy–tailed the process is.

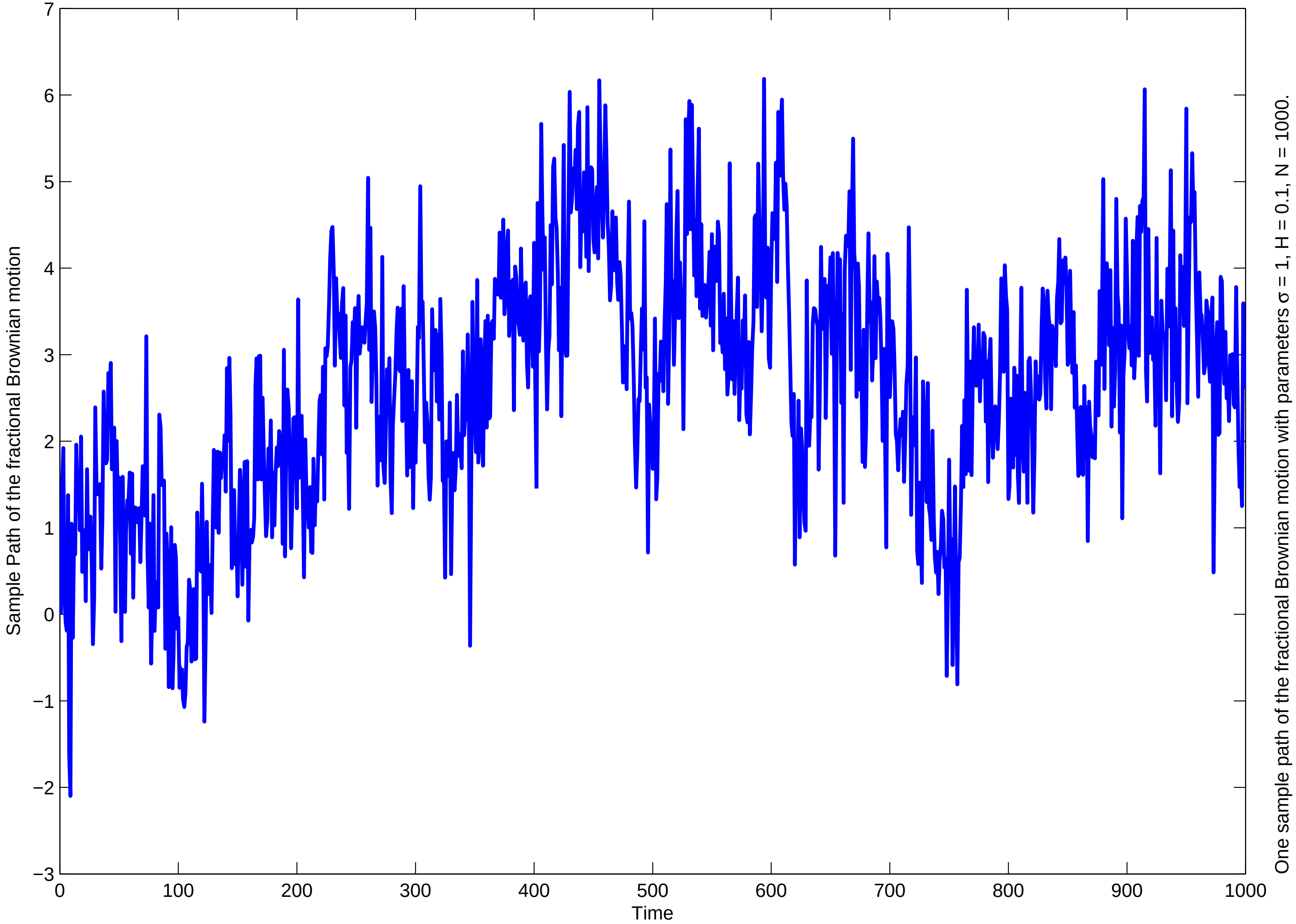
Fractional Brownian Motion

Definition 6.1
Let $0 < H \leq 1$*. A real–valued Gaussian process* $(B_H(t))_{t\geq 0}$ *is called* **fractional Brownian motion** *if* $\mathbb{E}[B_H(t)] = 0$ *and*

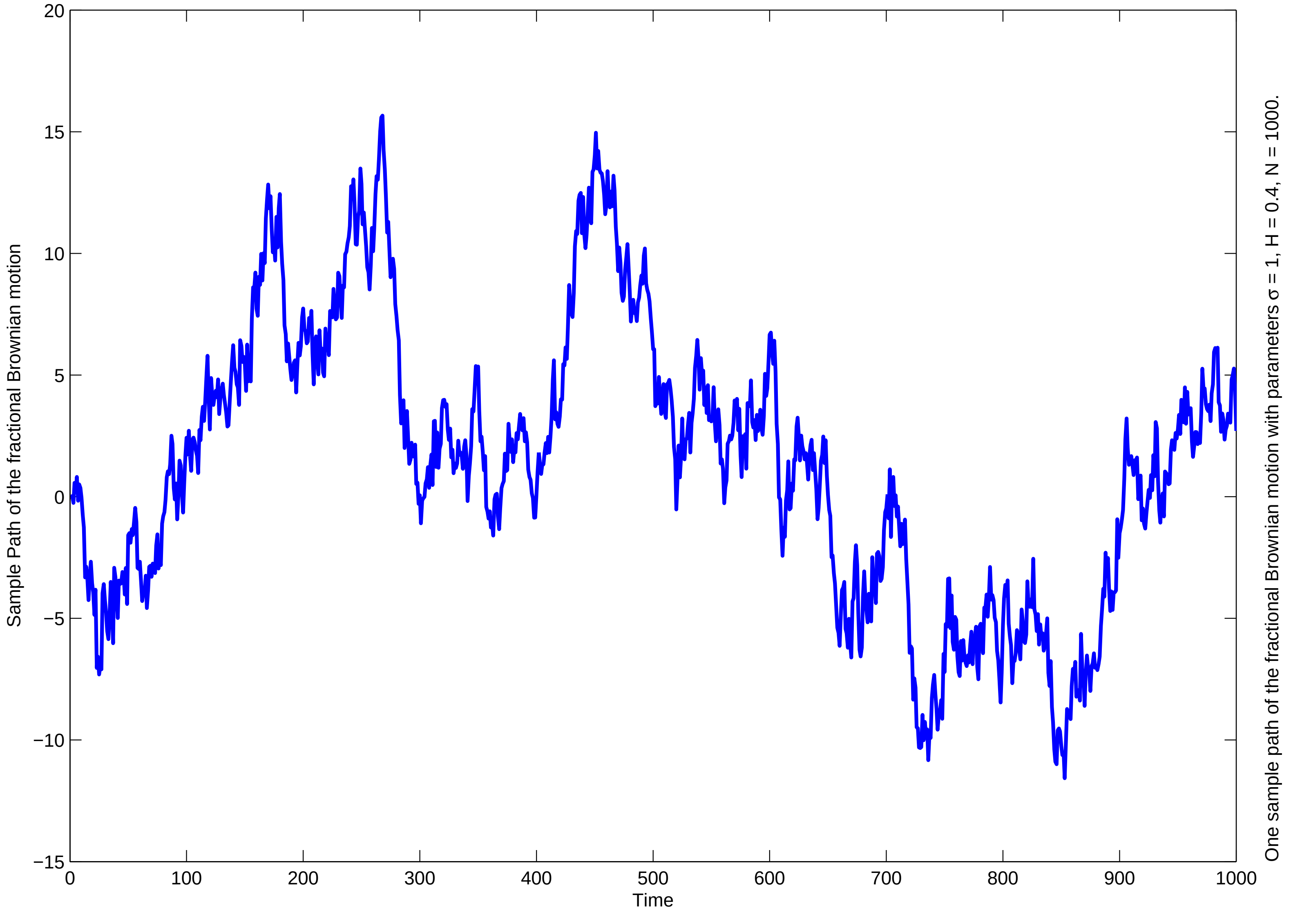
$$\mathbb{E}[B_H(t) \cdot B_H(s)] = \frac{1}{2}\left\{t^{2H} + s^{2H} - \mid t - s \mid^{2H}\right\} \cdot \mathbb{E}\left[B_H(1)^2\right].$$

- Note that the distribution of a Gaussian process is determined by its mean and covariance structure. Hence, the two conditions given in the above definition specify a unique Gaussian process.
- $\left(B_{1/2}(t)\right)_{t\geq 0}$ is a Brownian motion up to a multiplicative constant.

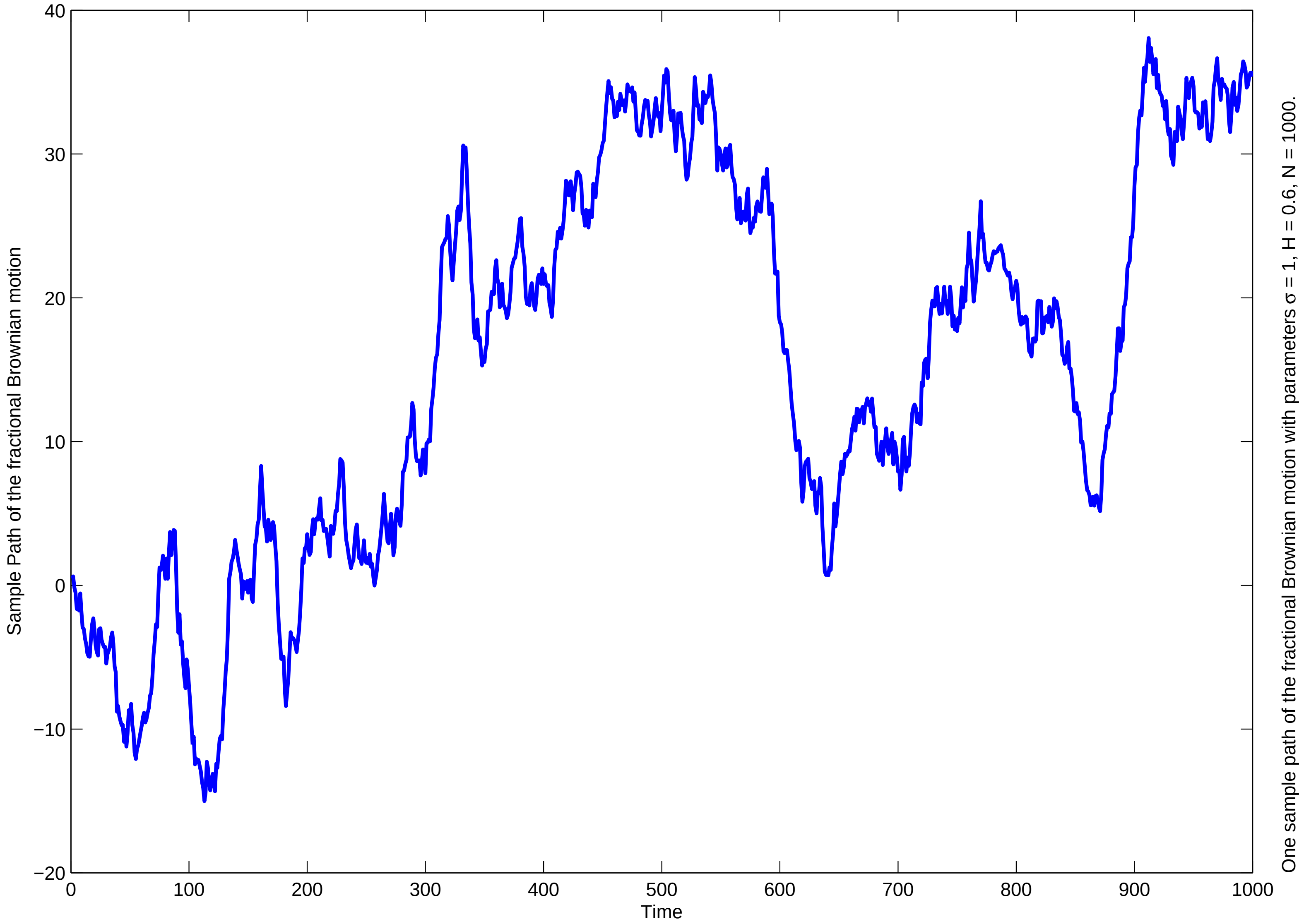
A Sample Path of the Fractional Brownian Motion with $H = 0.1$



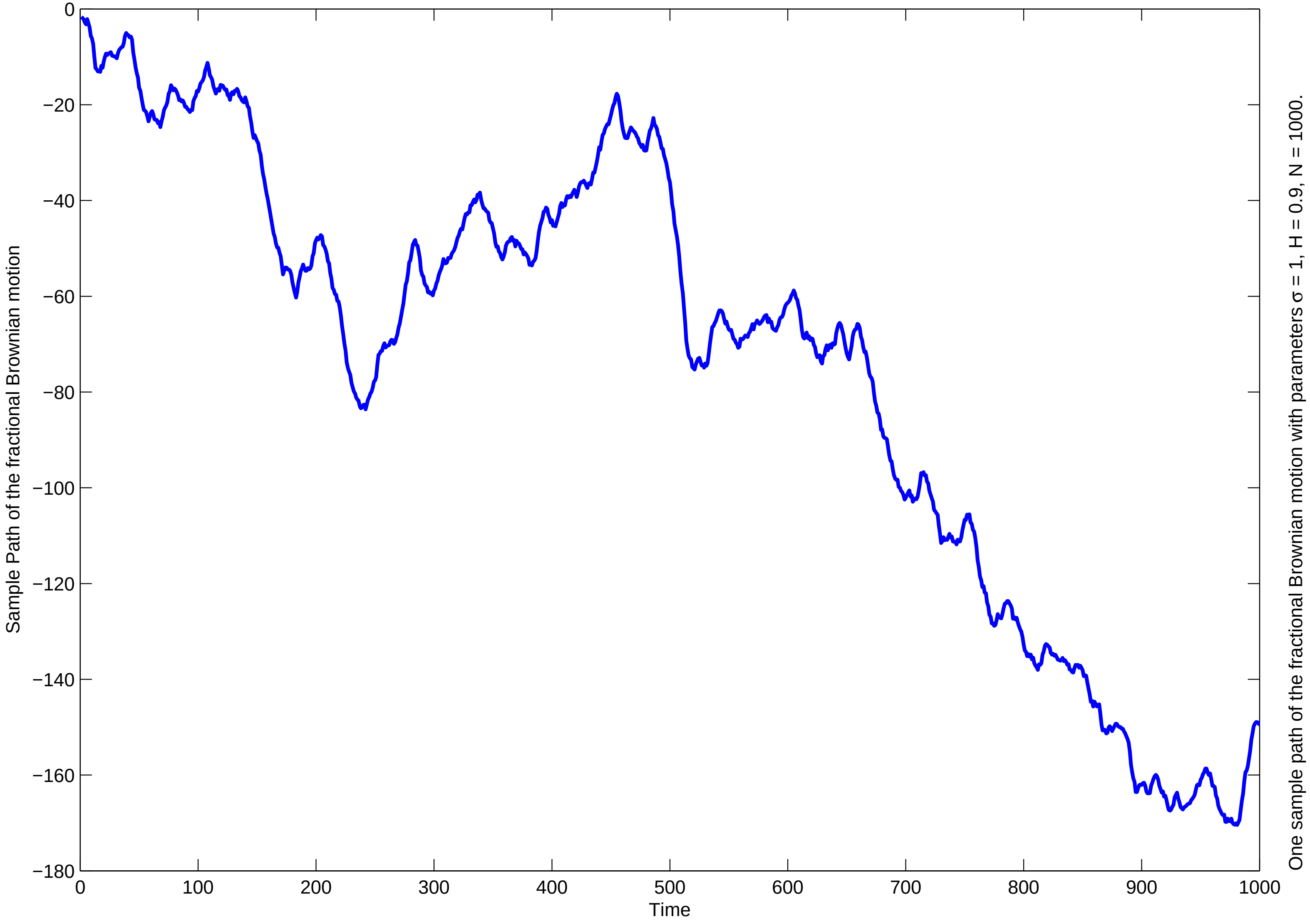
A Sample Path of the Fractional Brownian Motion with $H = 0.4$



A Sample Path of the Fractional Brownian Motion with $H = 0.6$



A Sample Path of the Fractional Brownian Motion with $H = 0.9$



Stable Lévy Processes

Definition 6.2

A (cadlag) stochastic process $(X_t)_{t\geq 0}$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *with values in* $\mathbb{R}^d$ *such that* $X_0 = 0$ *is called a* **Lévy process** *if it possesses the following properties:*

1. **Independent increments**: *for every increasing sequence of times* $t_0 \ldots t_n$, *the random variables* X_{t_0}, $X_{t_1} - X_{t_0}$, ..., $X_{t_n} - X_{t_{n-1}}$ *are independent.*
2. **Stationary increments**: *the distribution of* $X_{t+h} - X_t$ *does not depend on* t.
3. **Stochastic continuity**: $\forall \epsilon > 0$, $\lim_{h\to 0} \mathbb{P}\left(|X_{t+h} - X_t| \geq \epsilon\right) = 0$.

Definition 6.3

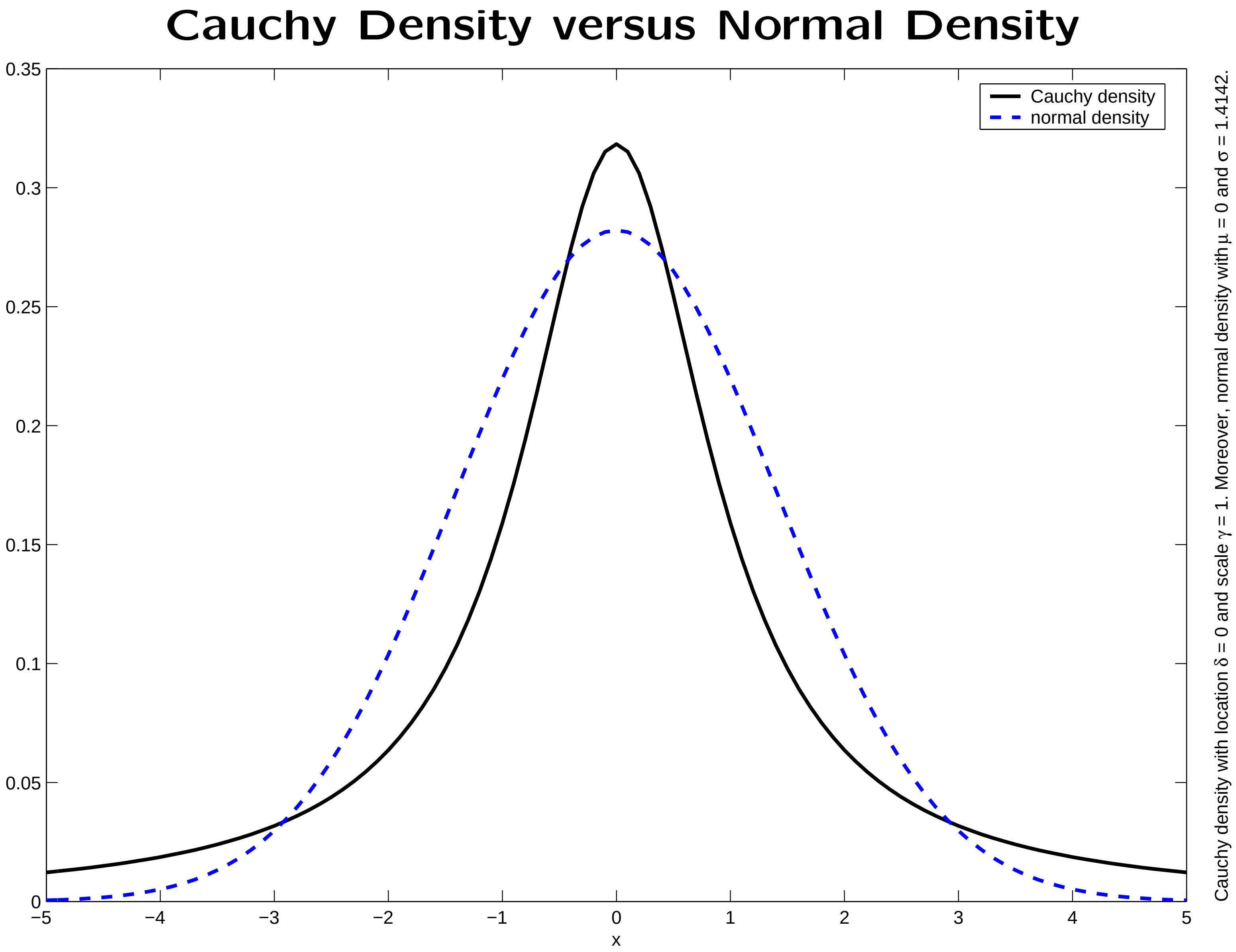
A probability measure μ *on* $\mathbb{R}^d$ *is called* **(strictly) stable**, *if for any* $a > 0$, *there exits* $b > 0$ *such that* $\widehat{\mu}(\theta)^a = \widehat{\mu}(b \cdot \theta)$ *for all* $\theta \in \mathbb{R}^d$.

Theorem 6.4
If μ on $\mathbb{R}^d$ is stable, there exits a unique $\alpha \in (0,2]$ such that $b = a^{1/\alpha}$. Such a μ is referred to as α–stable. When $\alpha = 2$, μ is a mean zero Gaussian probability measure.

Theorem 6.5
Suppose $(X_t)_{t \geq 0}$ is a Lévy process. Then $\mathcal{L}(X(1))$ is stable if and only if (X_t) is self–similar. The index α of stability and the exponent H of self–similarity satisfy $\alpha = \frac{1}{H}$.

- If Z_α is an $\mathbb{R}^d$–valued random variable with a α–stable distribution, $0 < \alpha < 2$, then for any $\gamma \in (0, \alpha)$, $\mathbb{E}\left[| Z_\alpha |^\gamma\right] < \infty$, but $\mathbb{E}\left[| Z_\alpha |^\alpha\right] = \infty$.

- Note that being heavy–tailed implies that the variance is infinite.


Cauchy Density versus Normal Density
Cauchy density
normal density
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−5
−4
−3
−2
−1
0
1
2
3
4
5
x
Cauchy density with location δ = 0 and scale γ = 1. Moreover, normal density with μ = 0 and σ = 1.4142.

7 Conclusion

The main results are that

- the scaling coefficient 0.5 has to be used very carefully for financial time series and

- there are substantial doubts about the self–similarity of the underlying processes of financial time series.

Concerning the scaling law, it is better to use a scaling law of 0.55 for the left quantile and a scaling law of 0.6 for the right quantile (the short positions) in order to be on the safe side.

It is important to keep in mind that these figures are only based on the (highly traded) DAX– and Dow Jones Index–stocks. Considering low traded stocks might yield even higher maximal scaling laws. These numbers should be set by the market supervision institutions like the SEC.

However, it is possible for the banks to reduce their Value at Risk–figures if they use the correct scaling law numbers. For instance, the Value at Risk–figure of a well diversified portfolio of Dow Jones Index–stocks would be reduced in this way about 12%, since it would have a scaling law of approximately 0.44.

8 Extensions

- Using bootstrap–methods in order to overcome the lack of data.

- Developing a test on self–similarity on the quantiles which overcomes the phenomena already described by Granger and Newbold.

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3. Gennardy Samorodnitsky and Murad S. Taqqu, *Stable Non–Gaussian Processes*, Chapman & Hall, London, 1994.