# Solving simultaneous linear equations using matrices

$$\begin{cases} 2x + 5y = 4 \\ x + 3y = 2 \end{cases}$$

$$\begin{cases} 3t + 2u = 1 \\ 4u - t = 5 \end{cases}$$

$$\begin{cases} 2p + 3q = 18 \\ 6p + 9q = 54 \end{cases}$$

$$\begin{cases} 6p - 3q + 3r = 0 \\ 3p + q - 6r = -10 \end{cases}$$

If $3x_1 + x_2 = 3a_1 + a_2 + 4$ and $5x_1 + 2x_2 = 5a_1 + 2a_2 - 3$, find expressions for $x_1$, $x_2$ in terms of $a_1, a_2$.

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$$\begin{cases} 2x + 5y = 4 \\ x + 3y = 2 \end{cases} \iff \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\det\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = 2\times 3 - 1\times 5 = 6-5 = 1 \neq 0$$

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}^{-1} = \frac{1}{1}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}}_{=I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12-10 \\ -4+4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\underline{x=2,\ y=0}$$

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$$\begin{cases} 3t + 2u = 1 \\ 4u - t = 5 \end{cases} \iff \begin{cases} 3t + 2u = 1 \\ -t + 4u = 5 \end{cases} \iff \underbrace{\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} t \\ u \end{pmatrix}}_{\underline{x}} = \underbrace{\begin{pmatrix} 1 \\ 5 \end{pmatrix}}_{\underline{v}}$$

$$A\underline{x} = \underline{v}$$

$$|A| = 3\times 4 - 2\times(-1)$$
$$= 12 + 2 = 14 \neq 0 \text{ so } A^{-1} \text{ exists.}$$

$$\underline{x} = A^{-1}\underline{v}$$

$$A^{-1} = \frac{1}{14}\begin{pmatrix} 4 & -2 \\ 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} t \\ u \end{pmatrix} = \underline{x} = A^{-1}\underline{v} = \frac{1}{14}\begin{pmatrix} 4 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \end{pmatrix}$$
$$= \frac{1}{14}\begin{pmatrix} 4 - 10 \\ 1 + 15 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} -6 \\ 16 \end{pmatrix}$$

$$t = \frac{-6}{14} = \underline{\underline{\frac{-3}{7}}}, \qquad u = \frac{16}{14} = \underline{\underline{\frac{8}{7}}}$$

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$$\begin{cases} 2p + 3q = 18 & \text{—①} \\ 6p + 9q = 54 & \text{—②} \end{cases} \iff \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 18 \\ 54 \end{pmatrix}$$

$$\xrightarrow{② - 3①} \begin{cases} 2p + 3q = 18 \\ 0 = 0 \end{cases}$$

$$\begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = 2 \times 9 - 3 \times 6 = 18 - 18 = 0$$

<u>either</u> 0 solutions

<u>OR</u> $\infty$ solutions

$$\xrightarrow{① \mapsto \frac{1}{2}①} \begin{cases} p + \frac{3}{2}q = 9 \\ 0 = 0 \end{cases}$$

augmented matrix

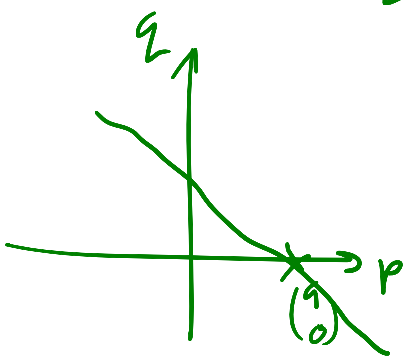
$$\left(\begin{array}{cc|c} 2 & 3 & 18 \\ 6 & 9 & 54 \end{array}\right) \xrightarrow{R2 \mapsto R2 - 3R1} \left(\begin{array}{cc|c} 2 & 3 & 18 \\ 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R1 \to \frac{1}{2}R1} \left(\begin{array}{cc|c} 1 & 3/2 & 9 \\ 0 & 0 & 0 \end{array}\right)$$

$$p = 9 - \frac{3}{2}q$$

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 9 - \frac{3}{2}q \\ q \end{pmatrix}$$

$$= \begin{pmatrix} 9 \\ 0 \end{pmatrix} + q \begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$$



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$$\begin{cases} 6p - 3q + 3r = 0 \\ 3p + q - 6r = -10 \end{cases}$$

$$\left(\begin{array}{ccc|c} 6 & -3 & 3 & 0 \\ 3 & 1 & -6 & -10 \end{array}\right)$$

$$\xrightarrow{R_1 \mapsto R_1 - 2R_2} \left(\begin{array}{ccc|c} 0 & -5 & 15 & 20 \\ 3 & 1 & -6 & -10 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 3 & 1 & -6 & -10 \\ 0 & -5 & 15 & 20 \end{array}\right)$$

$$\xrightarrow{R_2 \to -\frac{1}{5}R_2} \left(\begin{array}{ccc|c} 3 & 1 & -6 & -10 \\ 0 & 1 & -3 & -4 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left(\begin{array}{ccc|c} 3 & 0 & -3 & -6 \\ 0 & 1 & -3 & -4 \end{array}\right)$$

$$\xrightarrow{R_1 \mapsto \frac{1}{3}R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & -3 & -4 \end{array}\right)$$

$$\begin{cases} p \quad\;\; - r = -2 \\ \quad q - 3r = -4 \end{cases}$$

$$\begin{cases} p = -2 + r \\ q = -4 + 3r \end{cases}$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} -2+r \\ -4+3r \\ r \end{pmatrix} = \underline{\underline{\begin{pmatrix} -2 \\ -4 \\ 0 \end{pmatrix} + r\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}}}$$

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If $3x_1 + x_2 = 3a_1 + a_2 + 4$ and $5x_1 + 2x_2 = 5a_1 + 2a_2 - 3$, find expressions for $x_1$, $x_2$ in terms of $a_1, a_2$.

$$\begin{cases} 3x_1 + x_2 = 3a_1 + a_2 + 4 \\ 5x_1 + 2x_2 = 5a_1 + 2a_2 - 3 \end{cases} \iff \underbrace{\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\underline{x}} = \underbrace{\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}_{\underline{a}} + \underbrace{\begin{pmatrix} 4 \\ -3 \end{pmatrix}}_{\underline{b}}$$

$$\iff M\underline{x} = M\underline{a} + \underline{b}$$

$$\iff M\underline{x} - M\underline{a} = \underline{b}$$

$$\iff M(\underline{x} - \underline{a}) = \underline{b}$$

$$\iff \underline{x} - \underline{a} = M^{-1}\underline{b}$$

$$\iff \underline{x} = M^{-1}\underline{b} + \underline{a}$$

$$\iff \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ -3 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\iff \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 11 \\ -29 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$\det(M) = 3\times 2 - 1\times 5$
$= 6 - 5 = 1$
$\neq 0$

$$M^{-1} = \frac{1}{1}\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$$

$$x_1 = 11 + a_1$$
$$x_2 = -29 + a_2$$